Visualisation, Rendering and Animation

2 VO / 1 KU (2001-2004)

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Short podcast version 2020

3. Free-form Curves/Surfaces

Content

3. Parametric representation of curves and surfaces

- Free-form curves
  - Bézier-curves
  - Rational Bézier-curves
  - B-Splines
  - NURBS - industrial standard, Maya...
- Free-form surfaces

Motivation

- Construction, CAGD, CAD/CAM
  - Modeling of ship hulls, terminology
  - Design of cars and airplanes
- Computer Graphics
  - Simple modeling of smooth surfaces and solids
  - Definition of motion trajectories for animated objects

Three forms of expression

- ***Analytic*** **y = sqrt(r^2-x^2)**
- ***Implicit*** **$x^2 + y^2 = r^2$**
- ***Parametric*** **x = cos t; y = sin t; t $\in <0,2\pi>$** ***travel along the curve***
  - ***continuity - geometric G, parametric C***

Parametric Blending

- Numbers, a, b, 0.5*(a + b)
- Points A, B, 0.3*A + 0.7*B
- Rotations, 4-tuples, quaternions
- Curve construction as weighted sum of points
- Surfaces
- Images… brightness, contrast, saturation, sharpening… image analogies, SIGGRAPH 2001
- … Morphing, Caricatures … state spaces

Parametrically rep. curves

- *Euclidean plane/space E_2, E_3 {O, x, y, z}*
- *Parametric representation of a curve in E_3:*

$$\vec{p}(t) = \left(x(t) \quad y(t) \quad z(t)\right)^T \qquad \dot{\vec{p}}(t) \neq \vec{0}$$

Tangenta has direction vector: $\dot{\vec{p}}(t_0)$

- *Curvature:*

$$\kappa(t_0) = \frac{\left|\dot{\vec{p}}(t_0) \times \ddot{\vec{p}}(t_0)\right|}{\left|\dot{\vec{p}}(t_0)\right|^3}$$

Lagrange Interpolation

Given: Point set $\{a_0, \ldots a_n\}$ and appropriate parameter values $\{t_0 < \ldots < t_n\}$

Task: Curve through a_i for t_i

1. solution:

$$\vec{p}(t) = f_0(t)\vec{a}_0 + \ldots f_n(t)\vec{a}_n$$

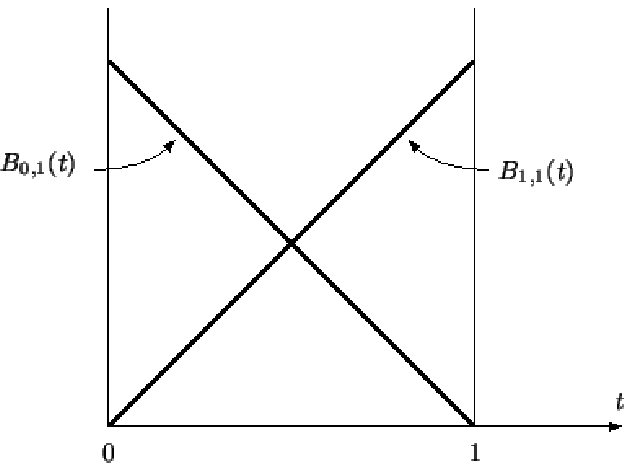
Lagrange -Polynomial:

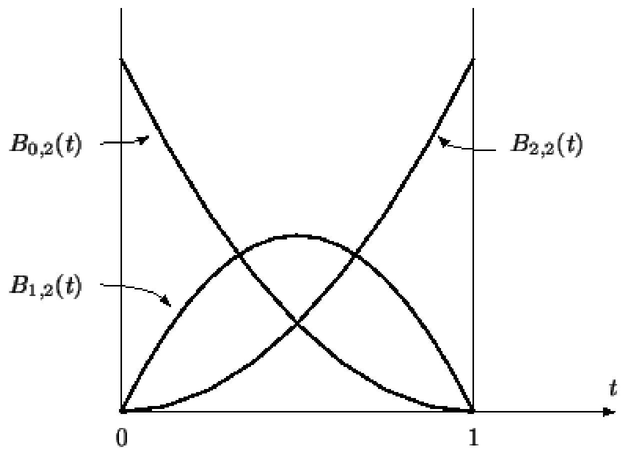
$$f_i(t) = \frac{(t-t_0)(t-t_1)\ldots(t-t_{i-1})(t-t_{i+1})\ldots(t-t_n)}{(t_i-t_0)(t_i-t_1)\ldots(t_i-t_{i-1})(t_i-t_{i+1})\ldots(t_i-t_n)}$$

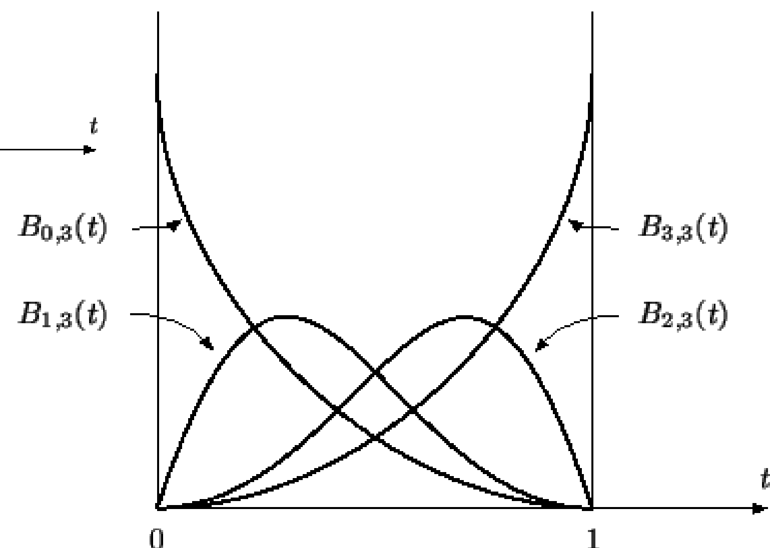
Bernstein Polynomials

- $$B_i^n(t) = \binom{n}{i}(1-t)^{n-i}\,t^i \qquad i = 0 \ldots n$$
- **Properties:**
  - **Polynomials of order n**
  - $$1 = \sum_{i=0}^{n} B_i^n(t)$$

Bernstein Polynomials: (C) Ken JOY







Next Properties

– $$\binom{n}{i} = \frac{n!}{(n-i)!\,i!} = \binom{n}{n-i}$$

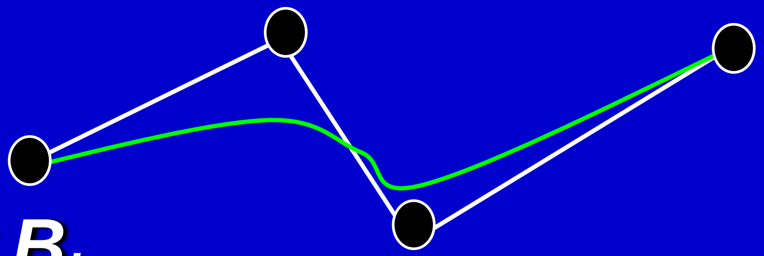
$$B^{n}_{n-i}\left(t^{*}\right) = B^{n}_{i}(t) = B^{n}_{i}\left(1-t^{*}\right) \qquad t^{*} = (1-t)$$

– ***{1, t, ... t^n} is basis for polynomials the same way is {B_0, ... B_n} another basis***

Bézier curve

- **Definition:** $$\vec{p}(t) = \sum_{i=0}^{n} B_i^n(t)\,\vec{b}_i \qquad t \in [0,1]$$

- **Basic notions:**
  - Base points B_i
  - B_i limited by base polygon
  - b_i position vector for B_i



Properties

- ***t = 0:***

$$B_0^n(0) = 1$$

$$B_i^n(0) = 0, \quad i = 1, \ldots n \quad \Rightarrow \vec{p}(0) = \vec{b}_0$$

- ***t = 1:***

$$B_n^n(1) = 1$$

$$B_i^n(1) = 0, i = 0, \ldots n-1 \quad \Rightarrow \vec{p}(1) = \vec{b}_n$$

- ***k-th derivative: depends on location t = 0 only of knot points B_0, ... B_k . Analogously for t = 1.***

DeCasteljau Algorithm

- ***Given:*** ***Base points*** $\{b_0, \ldots b_n\}$, t

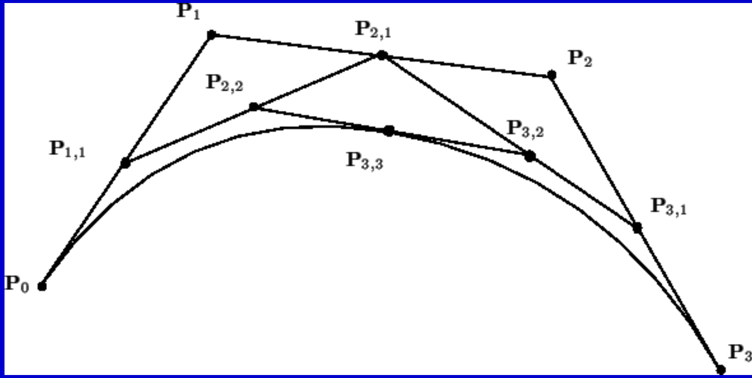


Fig. © by Ken Joy

- ***n iterations***

$$\vec{q}[i,0] = \vec{b}_i$$

$$\vec{q}[i, j+1] = (1-t)\vec{q}[i, j] + t\vec{q}[i+1, j]$$

$$\vec{q}[0, n] = \vec{x}(t)$$

- ***Version using one-dimensional array possible***

Parameter transformations

- $t \in [0, a] \qquad a \leq 1$

part of the original curve

- $t \in [0, a] \qquad a > 1$

original curve is the part - plus continuation

Spline curve

- <u>Def.:</u> „Spline curve“ consists of partial segments (Subsplines), combined by tangential or curvature preserving conditions
- <u>Example:</u> Bézier spline curve, binded from 2 Bézier curve pieces using the continuation.

Rational Bézier curves

- *Introduction of weights $\{w_0, \ldots w_n\}$ with the base points*
- *New form of the base polygon:*

$$\vec{b}_i \rightarrow \vec{B}_i = \begin{cases} \begin{pmatrix} \omega_i \\ \omega_i \vec{b}_i \end{pmatrix} & \omega_i \neq 0 \\ \begin{pmatrix} 0 \\ \vec{b}_i \end{pmatrix} & \omega_i = 0 \end{cases}$$

- *Representation via projection in the image plane*

Properties

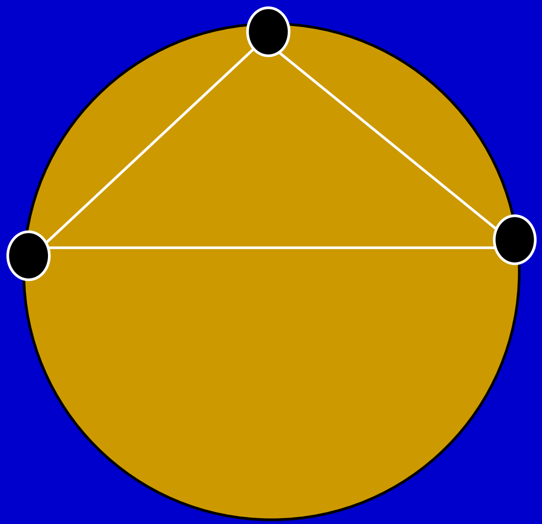
- $w_0 = \ldots w_n = 1$ -> standard Bézier
- $w_0 = \ldots w_n$ <> 0 -> rational Bézier

- Changing single weight:
  – w_i increase: the curve goes closer
  – w_i decrease: the curve goes far

- Modeling in higher dimensions, followed by projection

Special Case - Circle

- Bézier curves cannot represent!
- Suitable setting of weights works for rational Bézier curves



B-Splines

- ***Idea:*** ***Constant curvature setting of Bézier curves with Grad = 3.***
- ***Given:*** ***Basis point b_0-b_5 define 3 Bézier curves with n = 3, when the subintervals are known (Design parameter)***
- ***Then:*** ***The basis points of partial curves can be reconstructed.***

Additional Properties

- „Local Control“: Control points influence the curve in one position t
- Implication:
  - curve can be modified using one control point
  - 2 different tangents in one point possible!

Math Language Ruptures

- Elementary Arithmetics
- Synthetic Geometry
- Algebra
- <u>Analytic Geometry</u>
- Infinitesimal Calculus
- Iterative Geometry
- Predicate Calculus
- Set Theory

- (based on Kvasz´s epistemologic research, 1996)

Analytic Geometry

- Rene Descartes discovered the method how to assign to a given algebraic formula THE SHAPE.
- This visualization was so important that this new language was given a new name: analytic geometry.
- Constructive geometry using ruler and compass was difficult - for any object requires a specialised method and is limited to quadrics.
- Descartes method deconstructed each shape to points and enables us for constructing any shape POINT BY POINT.
- Therefore the curves are UNIVERSAL MODELING TOOL: car industry, flight simulations, caustics... Never ending story of applications.
  - (based on Kvasz´s epistemologic research, 1996)

Thank You...

… for Your attention.

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