PROCEEDINGS OF SYMPOSIUM ON COMPUTER GEOMETRY SCG ' 2011, VOLUME 20
pp. xy -xz

ON OSCULATING PSEUDO-CIRCLES OF CURVES IN THE PSEUDO-EUCLIDEAN PLANE [1]

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Abstract: The aim of the paper is to discuss some properties of the osculating pseudo-circles and the evolute of curves in the pseudo-Euclidean plane.

Key words: Pseudo-Euclidean plane, curve, osculating pseudo-circle, evolute

1. Introduction

The concept of osculating circle and evolute is well-known in differential geometry of the Euclidean plane. In this paper, we discuss some properties of these objects in the pseudo-Euclidean plane and compute them for selected curves.

The pseudo-Euclidean plane $\mathbf{E^{1,1}}$ is the real affine plane A^2 furnished with a non-singular indefinite quadratic form $q(\boldsymbol{x}) = \langle \boldsymbol{x}, \boldsymbol{x} \rangle$, where $\langle \, , \rangle$ denotes *the pseudo-scalar product*. This pseudo-scalar product can be expressed as $\langle \boldsymbol{x}, \, \boldsymbol{y} \rangle = x_1 y_1 - x_2 y_2$ in a suitable basis. *The length of a vector* $\boldsymbol{x}$ is defined as $|\boldsymbol{x}| = |q(\boldsymbol{x})|^{1/2}$. *Orthogonality* of vectors $\boldsymbol{x}, \boldsymbol{y}$ means $\langle \boldsymbol{x}, \, \boldsymbol{y} \rangle = 0$.

To any vector $\boldsymbol{x}$, we define its *sign* as $\operatorname{sgn} \boldsymbol{x} = \operatorname{sgn} (q(\boldsymbol{x}))$. We say that $\boldsymbol{x}$ is *a space-like vector* or *a time-like vector* or *a light-like vector* if $\operatorname{sgn} \boldsymbol{x} = 1$ or $\operatorname{sgn} \boldsymbol{x} = -1$ or $\operatorname{sgn} \boldsymbol{x} = 0$, respectively. We shall make use of *a perpendicularity operator* $\boldsymbol{x} \rightarrow \perp \boldsymbol{x}$ which assigns the vector $\perp \boldsymbol{x} = (\operatorname{sgn} \boldsymbol{x} \, x_2, \operatorname{sgn} \boldsymbol{x} \, x_1)$ to a vector $\boldsymbol{x} = (x_1, x_2)$.

According to the type of the tangent vector at a point of a curve, the point is said to be a space-like or a time-like or a light-like point of the curve. We exclude all light-like points from all curves.

In the pseudo-Euclidean plane we can consider parametrized curves similarly as in the Euclidean plane. Recall that a parametrization $P(t), t \in I$ is *a unit speed parametrization* if $|P'(t)| = 1$ for all t. Any regular curve in the pseudo-Euclidean plane (without light-like points) possesses a unit speed parametrization. At every regular (and not light-like) point $P(t)$ of a curve we have *the oriented Frenet frame* consisting of vectors

$$\boldsymbol{t}(t) = P'(t)/|P'(t)| \, , \quad \boldsymbol{n_{or}}(t) = \perp \boldsymbol{t}(t) \tag{1.1}$$

and *the oriented curvature*

$$k_{or}(t) = \det(P'(t), P''(t))/|P'(t)|^3 \tag{1.2}$$

see [1]. Moreover, it holds $\operatorname{sgn} \boldsymbol{n_{or}}(t) = \operatorname{sgn} k_{or}(t)$.

We refer to [1] for more details about curves in the pseudo-Euclidean plane.

[1] This paper has been supported by the projects VEGA 1/0730/09 and UK/274/2011.

2. Osculating pseudo-circles of curves in the pseudo-Euclidean plane

In the pseudo-Euclidean plane the role of circles play Euclidean equilateral hyperbolas with equation $(x_1 - s_1)^2 - (x_2 - s_2)^2 = \delta\rho^2$, where $S = (s_1, s_2)$ is the centre, $\rho > 0$ is the radius of the pseudo-circles and $\delta \in \{-1, 1\}$. If $\delta = 1$ or $\delta = -1$ then all points of the pseudo-circle are time-like or space-like hence, we speak about *time-like or space-like pseudo-circles*, respectively.

Proposition 1 and definition *At a non-inflexional and non-light-like point of a curve, there exists a unique pseudo-circle having at least three point contact with the curve at the common point. We call this pseudo-circle the osculating pseudo-circle of the curve at that point.*

Proof: Let $P(t)$ be a parametrized curve. Consider the contact function of the curve with a pseudo-circle given by $f(t) = \langle P(t) - S, P(t) - S\rangle - \delta r^2, \delta \in \{-1, 1\}$ (see [2], Chapter 8). The conditions for at least three point contact are $f(t) = 0, f'(t) = 0$ and $f''(t) = 0$. So we have equations:

$$\langle P(t) - S, P(t) - S\rangle - \delta r^2 = 0$$

$$2\langle P(t) - S, P'(t)\rangle = 0$$

$$2\langle P(t) - S, P''(t)\rangle + 2\langle P'(t), P'(t)\rangle = 0$$

From the second equation we have $P'(t) \perp P(t) - S$, so $P(t) - S = c\boldsymbol{n_{or}}$. Compute c from the third equation:

$$c = -\frac{\langle P'(t), P'(t)\rangle}{\langle \boldsymbol{n_{or}}, P''(t)\rangle}$$

From $\boldsymbol{n_{or}} = \perp \boldsymbol{t} = \perp \frac{P'(t)}{|\langle P'(t), P'(t)\rangle|^{1/2}}$ we have $c = -\frac{\langle P'(t), P'(t)\rangle |\langle P'(t), P'(t)\rangle|^{1/2}}{\langle \perp P'(t), P''(t)\rangle}$.

Let $\boldsymbol{a} = (a_1, a_2), \perp \boldsymbol{a} = \text{sgn}\,\langle \boldsymbol{a}, \boldsymbol{a}\rangle (a_2, a_1)$ and $\boldsymbol{b} = (b_1, b_2)$ be vectors. Compute $\langle \perp \boldsymbol{a}, \boldsymbol{b}\rangle$:

$$\langle \perp \boldsymbol{a}, \boldsymbol{b}\rangle = \text{sgn}\,\langle \boldsymbol{a}, \boldsymbol{a}\rangle \langle (a_2, a_1), (b_1, b_2)\rangle = -\text{sgn}\,\langle \boldsymbol{a}, \boldsymbol{a}\rangle \det(\boldsymbol{a}, \boldsymbol{b})$$

Using this, we find that

$$c = -\frac{\langle P'(t), P'(t)\rangle |\langle P'(t), P'(t)\rangle|^{1/2}}{-\text{sgn}\langle P'(t), P'(t)\rangle \det(P'(t), P''(t))} = \frac{|\langle P'(t), P'(t)\rangle|^{3/2}}{\det(P'(t), P''(t))} = \frac{1}{k_{or}}.$$

Finally, from the first equation we have that $c^2\langle \boldsymbol{n_{or}}, \boldsymbol{n_{or}}\rangle = \delta r^2$, i.e. $\delta = \text{sgn}\,\boldsymbol{n_{or}} = \text{sgn}\,k_{or}$, so the radius of the osculating pseudo-circle is $r = \frac{1}{|k_{or}|}$. Its center is expressed as follows:

$$S = P(t) - \frac{|\langle P'(t), P'(t)\rangle|^{3/2}}{\det(P'(t), P''(t))}\boldsymbol{n_{or}} = P(t) - \frac{1}{k_{or}(t)}\boldsymbol{n_{or}}(t)$$

Moreover if $\boldsymbol{\delta = 1}$ or $\boldsymbol{\delta = -1}$ then the osculating pseudo-circle is time-like or space-like pseudo-circle, respectively.

Proposition 2 (c.f. [4]) *Let $P(t) = (x(t), y(t))$ be a parametrized curve and $P_0 = P(t_0)$ a non-inflexional and non-light-like point of this curve. Let $P_1 = P(t_1)$ and $P_2 = P(t_2)$ be two different points of the curve "approaching" P_0 and such that $t_1 < t_0 < t_2$. The osculating* *pseudo-circle of the curve at* P_0 *is limit of pseudo-circles passing through points* P_0, P_1 *and* P_2.

Proof Let k be a pseudo-circle of equation k: $(x-a)^2-(y-b)^2-\delta r^2=0$. This pseudo-circle passes through points P_0, P_1 and P_2.
We define a function $g(t)= (x(t)-a)^2-(y(t)-b)^2-\delta r^2$. It is obvious that $g(t_0)=g(t_1)=g(t_2)=0$, while $t_1<t_0<t_2$. From Rolle´s theorem there exist points $P_3=P(t_3)$ and $P_4=P(t_4)$ such that $t_1<t_3<t_0<t_4<t_2$ and $g'(t_3)=g'(t_4)=0$. We get:

$$g'(t)= 2(x(t)-a)x'(t)-2(y(t)-b)y'(t)$$

$$g'(t_3)= 2(x(t_3)-a)x'(t_3)-2(y(t_3)-b)y'(t_3)=0,\ t_1<t_3<t_0$$

$$g'(t_4)= 2(x(t_4)-a)x'(t_4)-2(y(t_4)-b)y'(t_4)=0,\ t_0<t_4<t_2$$

We can use Rolle´s theorem one more time and we have that there exist a point $P_5=P(t_5)$ such that $t_3<t_5<t_4$ and $g^{(2)}(t_5)=0$. So we get:

$$g''(t_5)= 2\left[\left(x'(t_5)\right)^2+(x(t_5)-a)x''(t_5)\right]-2\left[\left(y'(t_5)\right)^2+(y(t_5)-b)y''(t_5)\right]$$

Since the limit of t_1 and t_2 is t_0, so it is also the limit of t_3, t_4, t_5. Therefore we have a system of three equations in three unknowns a, b and r.

$$(x(t_0)-a)^2-(y(t_0)-b)^2-\delta r^2=0$$

$$(x(t_0)-a)x'(t_0)-(y(t_0)-b)y'(t_0)=0$$

$$\left[\left(x'(t_0)\right)^2+(x(t_0)-a)x''(t_0)\right]-\left[\left(y'(t_0)\right)^2+(y(t_0)-b)y''(t_0)\right]=0$$

From the second and third equations follows:

$$a=x(t_0)-\frac{\left(\left(x'(t_0)\right)^2-\left(y'(t_0)\right)^2\right)}{x'(t_0)y''(t_0)-x''(t_0)y'(t_0)}y'(t_0) \tag{2.1}$$

$$b=y(t_0)-\frac{\left(\left(x'(t_0)\right)^2-\left(y'(t_0)\right)^2\right)}{x'(t_0)y''(t_0)-x''(t_0)y'(t_0)}x'(t_0) \tag{2.2}$$

These equations can be expressed in the following form:

$$a=x(t_0)-\frac{\langle P'(t_0),P'(t_0)\rangle}{\det\left(P'(t_0),P''(t_0)\right)}y'(t_0)$$

$$b=y(t_0)-\frac{\langle P'(t_0),P'(t_0)\rangle}{\det\left(P'(t_0),P''(t_0)\right)}x'(t_0)$$

From the expression of the oriented curvature (see [1])

$$k_{or}(t_0)=\frac{\det\left(P'(t_0),P''(t_0)\right)}{|\langle P'(t_0),P'(t_0)\rangle|^3}=\frac{\operatorname{sgn}\langle P'(t_0),P'(t_0)\rangle\det\left(P'(t_0),P''(t_0)\right)}{\langle P'(t_0),P'(t_0)\rangle|\langle P'(t_0),P'(t_0)\rangle|^{1/2}}$$

can a and b be expressed in the following form:

$$a=x(t_0)-\frac{1}{k_{or}(t_0)}\frac{\operatorname{sgn}\langle P'(t_0),P'(t_0)\rangle}{|\langle P'(t_0),P'(t_0)\rangle|^{1/2}}y'(t_0)$$

$$b = y(t_0) - \frac{1}{k_{or}(t_0)} \frac{\operatorname{sgn}\langle P'(t_0), P'(t_0)\rangle}{|\langle P'(t_0), P'(t_0)\rangle|^{1/2}} x'(t_0)$$

From the first equation of the system we have $r = \frac{1}{|k_{or}(t_0)|}$. Note that the vector

$$\left(\frac{\operatorname{sgn}\langle P'(t_0), P'(t_0)\rangle}{|\langle P'(t_0), P'(t_0)\rangle|^{1/2}} y'(t_0), \frac{\operatorname{sgn}\langle P'(t_0), P'(t_0)\rangle}{|\langle P'(t_0), P'(t_0)\rangle|^{1/2}} x'(t_0) \right)$$

is the oriented normal vector $\boldsymbol{n_{or}}(t_0)$ of our curve.
So the solution of this system of equations is the center and the radius of the osculating pseudo-circle at P_0:

$$S = P(t_o) - \frac{1}{k_{or(t_o)}} \boldsymbol{n_{or}}(t_0) \quad \text{and} \quad r = \frac{1}{|k_{or}(t_0)|}$$

Example 1 Let us consider a Euclidean circle $P(t) = (r\cos t, r\sin t)$, $t \in \langle -\frac{1}{4}\pi, \frac{7}{4}\pi \big)$. Points $P(t)$, $t \in \left(-\frac{1}{4}\pi, \frac{1}{4}\pi\right) \cup \left(\frac{3}{4}\pi, \frac{5}{4}\pi\right)$ are time-like, the points $P(t), t \in \left(\frac{1}{4}\pi, \frac{3}{4}\pi\right) \cup \left(\frac{5}{4}\pi, \frac{7}{4}\pi\right)$ are space-like and the points $P\left(-\frac{1}{4}\pi\right), P\left(\frac{1}{4}\pi\right), P\left(\frac{3}{4}\pi\right), P\left(\frac{5}{4}\pi\right)$ are light-like (see [1], Ex. 1). At space-like and time-like points of the circle, we have $\boldsymbol{t}, \boldsymbol{n_{or}}$ and k_{or} expressed as:

$$\boldsymbol{t} = \left(-\frac{\sin t}{|\cos 2t|^{1/2}}, \frac{\cos t}{|\cos 2t|^{1/2}} \right)$$

$$\boldsymbol{n_{or}} = \left(\frac{\cos t}{|\cos 2t|^{1/2}}, -\frac{\sin t}{|\cos 2t|^{1/2}} \right) \quad \text{if } t \in \left(\frac{1}{4}\pi, \frac{3}{4}\pi\right) \cup \left(\frac{5}{4}\pi, \frac{7}{4}\pi\right)$$

$$\boldsymbol{n_{or}} = \left(-\frac{\cos t}{|\cos 2t|^{1/2}}, \frac{\sin t}{|\cos 2t|^{1/2}} \right) \quad \text{if } t \in \left(-\frac{1}{4}\pi, \frac{1}{4}\pi\right) \cup \left(\frac{3}{4}\pi, \frac{5}{4}\pi\right)$$

$$k_{or} = \frac{1}{r|\cos 2t|^{3/2}}$$

So the center and the radius of the osculating pseudo-circle are:

$$S = (r\cos t, r\sin t) + r\cos 2t\,(\cos t, -\sin t) \quad \text{if } t \notin \left\{-\frac{1}{4}\pi, \frac{1}{4}\pi, \frac{3}{4}\pi, \frac{5}{4}\pi\right\}$$

$$r_c = r|\cos 2t|^{3/2}$$

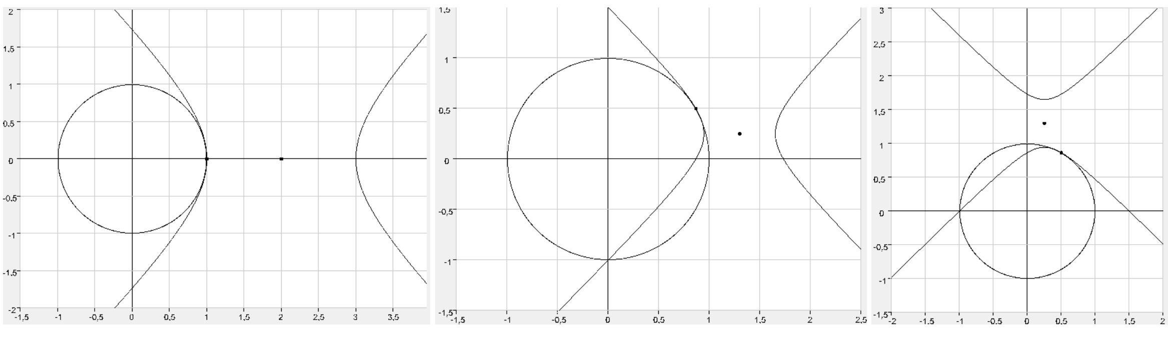

Fig. 1 Osculating pseudo-circles of a Euclidean circle at parameters

$$t = 0, t = \pi/6\,, t = \pi/3$$

Example 2 Let us consider a Euclidean ellipse $P(t) = (a\cos t\,, b\sin t), t \in \langle 0, 2\pi)$. Denote the function $a^2\sin^2 t - b^2\cos^2 t$ as $g(t)$. Points of the ellipse are time-like if $g(t) < 0$, they are space-like if $g(t) > 0$, and are light-like if $g(t) = 0$. In space-like and time-like points of the ellipse, we have $\boldsymbol{t}, \boldsymbol{n_{or}}$ and k_{or} expressed as follows:

$$\boldsymbol{t} = \left(-\frac{a\sin t}{|g(t)|^{1/2}}, \frac{b\cos t}{|g(t)|^{1/2}}\right)$$

$$\boldsymbol{n_{or}} = \left(\frac{b\cos t}{|g(t)|^{1/2}}, -\frac{a\sin t}{|g(t)|^{1/2}}\right) \quad \text{if } g(t) > 0$$

$$\boldsymbol{n_{or}} = \left(-\frac{b\cos t}{|g(t)|^{1/2}}, \frac{a\sin t}{|g(t)|^{1/2}}\right) \quad \text{if } g(t) < 0$$

$$k_{or} = \frac{ab}{|g(t)|^{3/2}}$$

So the center and the radius of the osculating pseudo-circle are:

$$S = (a\cos t\,, b\sin t) - \frac{g(t)}{ab}(b\cos t\,, -a\sin t) \quad \text{if } g(t) \neq 0$$

$$r = \frac{|g(t)\,|^{3/2}}{ab}$$

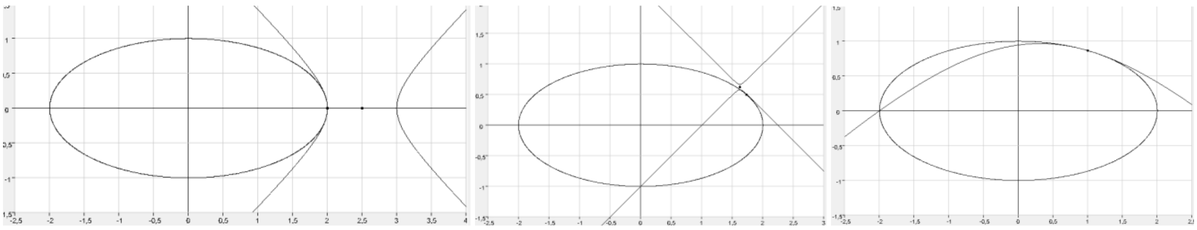

Fig. 2 Osculating pseudo-circles of an ellipse at parameters $t = 0, t = \pi/6\,, t = \pi/3$

Example 3 Let us consider a parabola $P(t) = (t^2/2p, t), t \in (-\infty, \infty)$. Points of the parabola are time-like if $|t| < p$, they are space-like if $|t| > p$, and are light-like if $|t| = p$. In space-like and time-like points of the parabola, we have:

$$\boldsymbol{t} = \left(\frac{t}{|t^2 - p^2|^{1/2}}, \frac{p}{|t^2 - p^2|^{1/2}}\right)$$

$$\boldsymbol{n_{or}} = \left(\frac{p}{|t^2 - p^2|^{1/2}}, \frac{t}{|t^2 - p^2|^{1/2}}\right) \quad \text{if } |t| > p$$

$$\boldsymbol{n_{or}} = \left(-\frac{p}{|t^2 - p^2|^{1/2}}, -\frac{t}{|t^2 - p^2|^{1/2}}\right) \quad \text{if } |t| < p$$

$$k_{or} = \frac{p^2}{|t^2 - p^2|^{3/2}}$$

As above, the center and the radius of the osculating pseudo-circle at any point $P(t)$ are:

$$S = (t^2/2p, t) + \frac{t^2 - p^2}{p}(1, t/p) \quad \text{if } |t| \neq p$$

$$r = \frac{|t^2 - p^2|^{3/2}}{p^2}$$

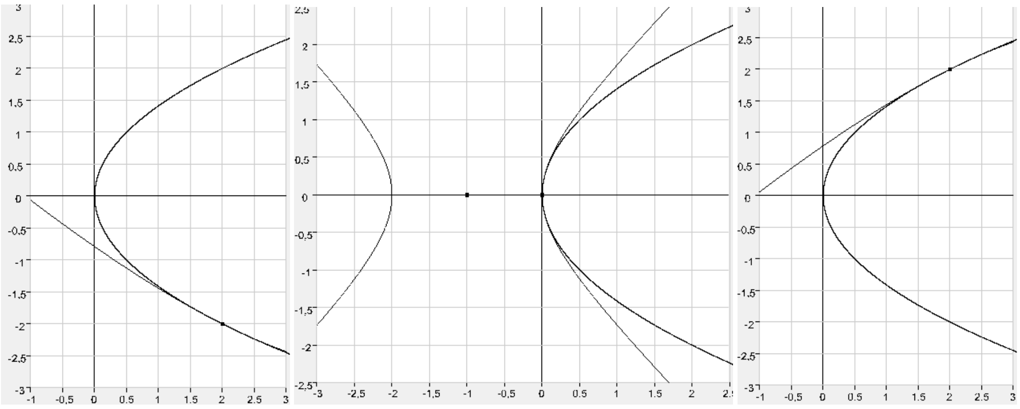

Fig. 3 Osculating pseudo-circles of a parabola at parameters $t = -2, t = 0, t = 2$

3. Evolute of a curve in the pseudo-Euclidean plane

As in the Euclidean plane, the evolute of a curve is the locus of all its centers of osculating circles (see [2], Chapter 8). From this we have the following definition:

Definition 1 *The evolute of a curve, with its inflexion and light-like points removed, is defined as the curve given by*

$$E(t) = P(t) - \frac{1}{k_{or}(t)}\boldsymbol{n_{or}}(t). \tag{3.1}$$

Note that the evolute of a curve can be expressed in coordinates using equations (2.1) and (2.2). This expression is much more convenient to use in computation of evolute.

Points of evolutes of curves in pseudo-Euclidean plane have similar properties to those in Euclidean case. We discuss these properties in the following proposition.

Recall that *a vertex* of a parametrized curve $P(t)$ is a point $P(t_0)$, in which the first derivate of curvature vanishes, i.e. $k'_{or}(t_0) = 0$. A point $P(t_0)$ is *an ordinary vertex* if $k'_{or}(t_0) = 0$ and $k''_{or}(t_0) \neq 0$. A regular point of a parametrized curve is a point where $P'(t) \neq 0$. Consider a unit-speed parametrization of the curve. Using Frenet formulas in the pseudo-Euclidean plane $\boldsymbol{t'} = k_{or}$, $\boldsymbol{n'_{or}} = k_{or}\boldsymbol{t}$ (see [1]) it is easy to show that for the first three derivates of evolute it holds:

$$E'(t) = \frac{k'_{or}(t)}{k^2_{or}(t)}\boldsymbol{n_{or}}(t) \tag{3.2}$$

$$E''(t) = \frac{k'_{or}(t)}{k_{or}(t_0)}\boldsymbol{t}(t) + \left(\frac{k''_{or}(t)}{k^2_{or}(t)} - 2\frac{\left(k'_{or}(t)\right)^2}{k^3_{or}(t)}\right)\boldsymbol{n_{or}}(t) \tag{3.3}$$

$$E'''(t) = \left(2\frac{k''_{or}(t)}{k_{or}(t)} - 3\frac{\left(k'_{or}(t)\right)^2}{k^2_{or}(t)}\right)\boldsymbol{t}(t) + \\ + \left(k'_{or}(t) + \frac{k'''_{or}(t)}{k^2_{or}(t)} - 6\frac{k'_{or}(t)k''_{or}(t)}{k^3_{or}(t)} + 6\frac{\left(k'_{or}(t)\right)^3}{k^4_{or}(t)}\right)\boldsymbol{n_{or}}(t) \quad (3.4)$$

Proposition 3 *Let $P(t)$ be a parametrized curve and $E(t)$its evolute.*

a) A point $E(t_0)$ of the evolute is a regular point if and only if the point $P(t_0)$ is not a vertex of the given curve.
b) If the point $P(t_0)$ is an ordinary vertex of the curve then the point $E(t_0)$ is a cusp of the first kind of the evolute.

Proof a) The assertion follows immediately from the formula (3.2).
b) From a) we have that if $P(t_o)$ is a vertex of the curve, $E(t_0)$ is a singular point of evolute. From (3.3) and (3.4) we have the second and the third derivate of the evolute in its ordinary vertex expressed as:

$$E''(t_0) = \frac{k''_{or}(t_0)}{k^2_{or}(t_0)}\boldsymbol{n_{or}}(t_0) \neq 0$$

$$E'''(t_0) = 2\frac{k''_{or}(t_0)}{k^2_{or}(t_0)}\boldsymbol{t}(t_0) + \frac{k'''_{or}(t_0)}{k^2_{or}(t_0)}\boldsymbol{n_{or}}(t_0) \neq 0$$

We see that the vectors $E''(t_0)$ and $E'''(t_0)$ are linearly independent, so $E(t_0)$ is a cusp of the first kind of the evolute.

Example 4 Example 1 shows that the evolute of a Euclidean circle $P(t) = (r\cos t, r\sin t), t \in \langle -\frac{1}{4}\pi, \frac{7}{4}\pi \big)$ is expressed as:

$$E(t) = (r\cos t, r\sin t) + r\cos 2t\,(\cos t, -\sin t) \qquad \text{if } t \notin \left\{-\frac{1}{4}\pi, \frac{1}{4}\pi, \frac{3}{4}\pi, \frac{5}{4}\pi\right\}$$

In coordinates we have the following expression of the evolute of the Euclidean circle:

$$x(t) = 2r\cos^3 t, y(t) = 2r\sin^3 t$$

It is obvious that this curve is the image of the astroid $Q(t) = (\cos^3 t, \sin^3 t)$ in the affine transformation $x^\sim = 2rx$ and $y^\sim = 2ry$.

Example 5 Example 2 shows that the evolute of a Euclidean ellipse $P(t) = (a\cos t, a\sin t), t \in \langle 0, 2\pi)$ is expressed as:

$$E(t) = (a\cos t, b\sin t) - \frac{a^2\sin^2 t - b^2\cos^2 t}{ab}(b\cos t, -a\sin t) \qquad \text{if } a^2\sin^2 t - b^2\cos^2 t \neq 0$$

In coordinates we have the following expression of the evolute of the ellipse:

$$x(t) = \frac{a^2 + b^2}{a}\cos^3 t, \qquad y(t) = \frac{a^2 + b^2}{b}\sin^3 t$$

It is obvious that this curve is the image of the astroid $Q(t) = (\cos^3 t, \sin^3 t)$ in the affine transformation $x^\sim = \frac{a^2+b^2}{a}x$ and $y^\sim = \frac{a^2+b^2}{b}y$.

Example 6 Example 3 shows that the evolute of a parabola $P(t) = (t^2/2p, t), t \in (-\infty, \infty)$ is:

$$E(t) = \left(\frac{t^2}{2p}, t\right) + \frac{t^2 - p^2}{p}\left(1, \frac{t}{p}\right) \quad \text{if } |t| \neq p$$

In coordinates we have the following expression of the evolute of the parabola:

$$x(t) = \frac{3}{2p}t^2 - \frac{p}{2}, \qquad y(t) = \frac{1}{p^2}t^3$$

It is obvious that this curve is the image of semi-cubical parabola $Q(t) = (t^2, t^3)$ in the affine transformation $x^{\sim} = \frac{3}{2p}x - \frac{p}{2}$ and $y^{\sim} = \frac{1}{p^2}y$.

Remark 1 From Example 5 and Example 6, it is obvious that evolutes of a parabola and an ellipse in the pseudo-Euclidean plane look similar to the ones in the classic Euclidean plane. On the other hand, the evolute of the Euclidean circle from Example 4 is strongly different to that in the Euclidean plane. While in the Euclidean plane this evolute degenerates to a single point, in the pseudo-Euclidean case it is an actual curve.

Remark 2 Another very interesting property of evolutes is that a curve (without light-like points) does not intersect its evolute, see [3], Prop 3.3. As well-known, this does not hold in the Euclidean plane because every ellipse intersects its evolute, there.

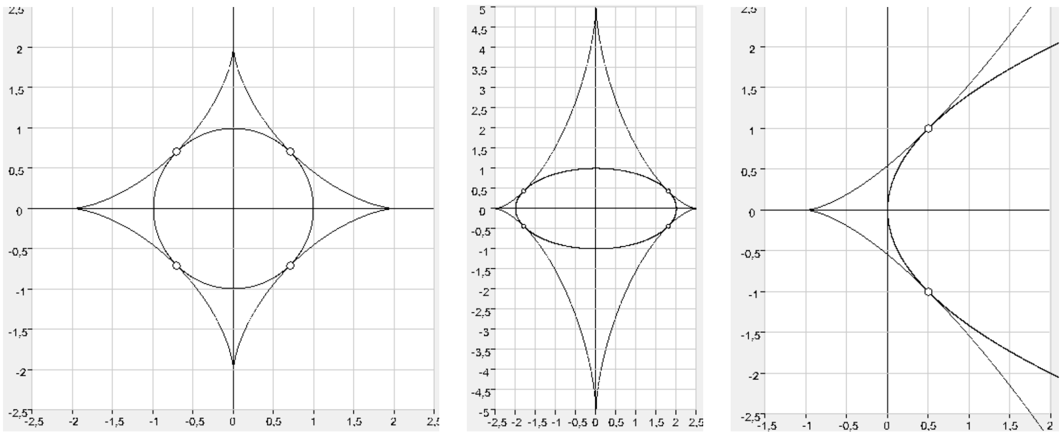

Fig. 4 Evolutes of a Euclidean circle, a Euclidean ellipse and a parabola

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