PROCEEDINGS OF SYMPOSIUM ON COMPUTER GEOMETRY SCG' 2012, VOLUME 21
pp. 0 - 0

ON SINGULARITIES OF EVOLUTES OF CURVES IN THE PSEUDO-EUCLIDEAN PLANE

Viktória Bakurová [1]

Comenius University, Faculty of Mathematics, Physics and Informatics
Mlynská dolina, 842 48 Bratislava, Slovak Republic,
e-mail: viktoria.bakurova@fmph.uniba.sk

Abstract. The aim of the paper is to classify singular points of arbitrary order of evolutes of curves in the pseudo-Euclidean plane. We point out a close connection of such singularities to higher-order vertices of base curves.

Keywords: Pseudo-Euclidean plane, curve, singular point, evolute.

1. Introduction

The concept of evolute is well-known in differential geometry of the Euclidean plane. In this paper, we discuss some properties of this object and classify singularities of these curves in the pseudo-Euclidean plane.

The pseudo-Euclidean plane $\mathbf{E^{1,1}}$ is the real affine plane $\mathbf{A^2}$ furnished with a nonsingular indefinite quadratic form $q(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x} \rangle$, where $\langle , \rangle$ denotes the pseudo-scalar product. This pseudo-scalar product can be expressed as $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 - x_2 y_2$ in a suitable basis.

We say that $\mathbf{x}$ is a space-like vector or a time-like vector or a light-like vector if $\operatorname{sgn} \mathbf{x} = 1$ or $\operatorname{sgn} \mathbf{x} = -1$ or $\operatorname{sgn} \mathbf{x} = 0$, respectively, where $\operatorname{sgn} \mathbf{x} = \operatorname{sgn} q(\mathbf{x})$. We shall make use of a perpendicularity operator $\mathbf{x} \to \perp \mathbf{x}$ which assigns the vector $\perp \mathbf{x} = (\operatorname{sgn} \mathbf{x}\, x_2, \operatorname{sgn} \mathbf{x}\, x_1)$ to a vector $\mathbf{x} = (x_1, x_2)$.

According to the type of the tangent vector at a point of a curve, the point is said to be a space-like or a time-like or a light-like point of the curve. We exclude all light-like points from all curves.

In the pseudo-Euclidean plane we can consider parametrized curves similarly as in the Euclidean plane. At every regular (and not light-like) point $P(t)$ of a curve we have the oriented Frenet frame consisting of vectors

$$\mathbf{t}(t) = \frac{P'(t)}{|P'(t))|}, \qquad \mathbf{n}(t) = \perp \mathbf{t}(t)$$

and the oriented curvature

$$k(t) = \frac{\det(P'(t), P''(t))}{|\langle P'(t), P'(t) \rangle|^{\frac{3}{2}}}.$$

Moreover, for a curve expressed in a unit speed parametrization it holds

$$\mathbf{t}'(s) = k(s)\mathbf{n}(s)$$
$$\mathbf{n}'(s) = k(s)\mathbf{t}(s).$$

We refer to [1] and [2] for more details about curves in the pseudo-Euclidean plane.

[1] This project has been supported by the project UK/371/2012.

2. Evolute of a curve in the pseudo-Euclidean plane

As in the Euclidean plane, the evolute of a curve in the pseudo-Euclidean plane is the locus of all its centers of osculating circles (see [4], Chapter 8). From this we have the following definition:

Definition 1. *The evolute of a curve, with its inflexion and light-like points removed, is defined as the curve given by*

$$E(t) = P(t) - \frac{1}{k(t)}\mathbf{n}(t). \tag{1}$$

Recall that a vertex of a parametrized curve $P(t)$ is a point $P(t_0)$, in which the first derivate of curvature vanishes, i.e. $k'(t_0) = \mathbf{0}$. A point $P(t_0)$ is an ordinary vertex if $k'(t_0) = \mathbf{0}$ and $k''(t_0) \neq \mathbf{0}$. A vertex of a curve is a vertex of order l if the first l derivates of curvature vanishes, i.e. $k'(t_0) = \mathbf{0}, \ldots, k^{(l)}(t_0) = \mathbf{0}$. A vertex of a curve is a vertex of order exactly l if it is a vertex of order l and in addition it holds $k^{(l+1)}(t_0) \neq \mathbf{0}$.

The next lemma shows us, how the first four derivates of evolute are expressed:

Lemma 1. *Consider a unit-speed parametrization of the curve. For the first four derivates of evolute of this curve it holds:*

$$\begin{aligned}
E^{(1)} &= \frac{k'}{k^2}\mathbf{n} \\
E^{(2)} &= \frac{k'}{k}\mathbf{t} + \left(\frac{k''}{k^2} - 2\frac{(k')^2}{k^3}\right)\mathbf{n} \\
E^{(3)} &= \left(2\frac{k''}{k} - 3\frac{(k')^2}{k^2}\right)\mathbf{t} + \left(k' + \frac{k'''}{k^2} - 6\frac{k'k''}{k^3} + 6\frac{(k')^3}{k^4}\right)\mathbf{n} \\
E^{(4)} &= \left(3\frac{k'''}{k} - 14\frac{k'k''}{k^2} + 12\frac{(k')^3}{k^3} + k'k\right)\mathbf{t} \\
&\quad + \left(3k'' + \frac{k''''}{k^2} - 3\frac{(k')^2}{k} - 8\frac{k'k'''}{k^3} - 6\frac{(k'')^2}{k^3} + 36\frac{(k')^2k''}{k^4} - 24\frac{(k')^4}{k^5}\right)\mathbf{n}
\end{aligned}$$

Proof. The assertion of the lemma can be easily shown using Frenet formulas in the pseudo-Euclidean plane. □

To obtain an adequate expression for derivative $E^{(n)}, n > 4$, we use the class $P_{n,1}(k)$ of functions from [3]. We summarize briefly basic facts on that class.

Let P_n be the real vector space of all polynomial functions $\varphi(x_0, \ldots, x_n)$ and let $P_{n,1}$ denotes its vector-subspace consisting of functions $\varphi(x_0, \ldots, x_n)$ with the property $\varphi(x_0, 0, \ldots, 0) = 0$ for every $x_0 \in \mathbb{R}$.

To the curvature $k(t)$ of the curve $P(t)$ defined on an interval $I \subseteq \mathbb{R}$ and to any integer $n \geq 0$ we assigning the $(n+1)$-tuple of derivatives $(k^{(0)}, \ldots, k^{(n)})$ understood as a function $I \to \mathbb{R}^{n+1}$. To every function $\varphi \in P_n$ or $\varphi \in P_{n,1}$ we get a one-variable function $\varphi(k^{(0)}, \ldots, k^{(n)}) : I \to \mathbb{R}, t \to \varphi(k^{(0)}(t), \ldots, k^{(n)}(t))$.

We denote the set of all such functions as $P_n(k)$ or $P_{n,1}(k)$, respectively. The most important properties of the classes $P_n(k)$ and $P_{n,1}(k)$ are following, see [3], Proposition 1. For every $n, m \geq 1$ it holds:

- if $\sigma \in P_{n,1}(k), \tau \in P_m(k)$ then $\sigma\tau \in P_{n,1}(k)$
- if $\sigma \in P_{n,1}(k), \tau \in P_{m,1}(k)$ then $\sigma + \tau, \sigma\tau \in P_{\max(n,m),1}(k)$
- $d(P_{n,1}(k)) \subseteq P_{n+1,1}(k)$ where $d : \sigma \to \sigma'$ is the derivate operator

Note that the last property follows from the chain rule, particularly.

The next theorem is a generalization of the previous lemma and it shows the expression for the derivation of arbitrary order of the evolute of a curve.

Theorem 2. *Let $E(t)$ be an evolute of a curve $P(t)$ expressed in unit speed parametrization. To every $n \geq 4$ there exist polynomial functions $\alpha_n \in P_{n-2,1}$ and $\beta_n \in P_{n-1,1}$ such that it holds:*

$$E^{(n)} = \{\left((n-1)k^n k^{(n-1)} + \alpha_n(k^{(0)}, \ldots, k^{(n-2)})\right) \mathbf{t} + \left(k^{n-1}k^{(n)} + \beta_n(k^{(0)}, \ldots, k^{(n-1)})\right) \mathbf{n}\} \frac{1}{k^{n+1}}.$$

Proof. We use induction over n. For $n = 4$, the assertion is true in virtue of Lemma 1. Let the formula holds for n. Let us denote $a_n = (n-1)k^n k^{(n-1)} + \alpha_n(k^{(0)}, \ldots, k^{(n-2)})$ and $b_n = k^{n-1}k^{(n)} + \beta_n(k^{(0)}, \ldots, k^{(n-1)})$ and differentiate:

$$\begin{aligned}
E^{(n+1)} &= \left\{ (a_n \mathbf{t} + b_n \mathbf{n}) \frac{1}{k^{n+1}} \right\}' \\
&= (a_n' \mathbf{t} + a_n k \mathbf{n} + b_n' \mathbf{n} + b_n k \mathbf{t}) \frac{1}{k^{n+1}} + (a_n \mathbf{t} + b_n \mathbf{n}) \left(\frac{1}{k^{n+1}} \right)' \\
&= (a_n' \mathbf{t} + a_n k \mathbf{n} + b_n' \mathbf{n} + b_n k \mathbf{t}) \frac{1}{k^{n+1}} + (a_n \mathbf{t} + b_n \mathbf{n}) \left(\frac{(-1)(n+1)k^{(1)}}{k^{n+2}} \right) \\
&= \{\left(k a_n' + k^2 b_n - (n+1)k^{(1)} a_n\right) \mathbf{t} + \left(k^2 a_n + k b_n' - (n+1)k^{(1)} b_n\right) \mathbf{n}\} \frac{1}{k^{n+2}}
\end{aligned}$$

Compute the following expression:

$$\begin{aligned}
k a_n' + k^2 b_n - (n+1)k^{(1)} a_n &= n(n-1)k^n k^{(1)} k^{(n-1)} + (n-1)k^{n+1}k^{(n)} + k\gamma_n(k^{(0)}, \ldots, k^{(n-1)}) \\
&+ k^{n+1}k^{(n)} + k^2\beta_n(k^{(0)}, \ldots, k^{(n-1)}) \\
&- (n-1)(n+1)k^n k^{(1)} k^{(n-1)} - (n-1)k^{(1)}\alpha_n(k^{(0)}, \ldots, k^{(n-2)}) \\
&= n k^{n+1} k^{(n)} + \alpha_{n+1}(k^{(0)}, \ldots, k^{(n-1)}),
\end{aligned}$$

where $\alpha_{n+1}(k^{(0)}, \ldots, k^{(n-1)}) = n(n-1)k^n k^{(1)} k^{(n-1)} + k\gamma_n(k^{(0)}, \ldots, k^{(n-1)}) + k^2\beta_n(k^{(0)}, \ldots, k^{(n-1)}) + (n-1)(n+1)k^n k^{(1)} k^{(n-1)} - (n-1)k^{(1)}\alpha_n(k^{(0)}, \ldots, k^{(n-2)})$ denotes polynomial function of the class $P_{n-1,1}$.

Compute now the following expression:

$$\begin{aligned}
k^2 a_n + k b_n' - (n+1)k^{(1)} b_n &= (n-1)k^{n+2}k^{(n-1)} + k^2\alpha_n(k^{(0)}, \ldots, k^{(n-2)}) \\
&+ (n-1)k^{n-1}k^{(1)}k^{(n)} + k^n k^{(n+1)} + \delta_n(k^{(0)}, \ldots, k^{(n)}) \\
&- (n+1)k^{n-1}k^{(1)}k^{(n)} - (n+1)k^{(1)}\beta_n(k^{(0)}, \ldots, k^{(n-1)}) \\
&= k^n k^{(n+1)} + \beta_{n+1}(k^{(0)}, \ldots, k^{(n)}),
\end{aligned}$$

where $\beta_{n+1}(k^{(0)},\ldots,k^{(n)}) = (n-1)k^{n+2}k^{(n-1)} + k^2\alpha_n(k^{(0)},\ldots,k^{(n-2)}) + (n-1)k^{n-1}k^{(1)}k^{(n)} + \delta_n(k^{(0)},\ldots,k^{(n)}) - (n+1)k^{n-1}k^{(1)}k^{(n)} - (n+1)k^{(1)}\beta_n(k^{(0)},\ldots,k^{(n-1)})$ denotes polynomial function of the class $P_{n,1}$.

Using these expressions we have:

$$E^{(n+1)} = \{\left(nk^{n+1}k^{(n)} + \alpha_{n+1}(k^{(0)},\ldots,k^{(n-1)})\right)\mathbf{t} + \left(k^n k^{(n+1)} + \beta_{n+1}(k^{(0)},\ldots,k^{(n)})\right)\mathbf{n}\}\frac{1}{k^{n+2}}.$$

The theorem is proved. □

3. Classification of singularities of evolutes in the pseudo-Euclidean plane

In this section we discuss singularities of evolute of the curve in the pseudo-Euclidean plane. We classify singular points of the evolute according to the order of its singularity.

Let us remind that a singular point of a curve $P(t)$ is a point $P(t_0)$ for which it holds $P^{(1)}(t_0) = \mathbf{0}$. We say that a point $P(t_0)$ is a singular point of order n, $n \geq 1$ of a curve $P(t)$ only and only if $P^{(1)}(t_0) = \mathbf{0}, \ldots, P^{(n)}(t_0) = \mathbf{0}$. A point $P(t_0)$ is a singular point of order exactly n, $n \geq 1$ if $P^{(1)}(t_0) = \mathbf{0}, \ldots, P^{(n)}(t_0) = \mathbf{0}$ and $P^{(n+1)}(t_0) \neq \mathbf{0}$. For the classification of singular points of a curve we need the next commonly used lemma:

Lemma 3. *Let $P(t)$ be a curve and let $P(t_0)$ be a singular point of order exatly $k, k > 1$.*

1. *If k is odd then the point $P(t_0)$ is an insignificantly singular point of the curve.*
2. *If k is even then the point $P(t_0)$ is a cusp of the curve.*
3. *Let k be even and let l be the lowest natural number so that $P^{(k)}(t_0)$ and $P^{(l)}(t_0)$ are linearly independent. Then the point $P(t_0)$ is a cusp of the first kind for l odd and a cusp of the second kind for l even.*

Let $P(t_0)$ be a vertex of a curve $P(t)$ of order exactly l. The next theorem shows us that the corresponding point $E(t_0)$ of the evolute of this curve is a singular point of order exactly l.

Theorem 4. *Let $P(t_0)$ be a vertex of a curve $P(t)$ of order exactly l. For the first l derivates of the evolute it holds that $E^{(1)}(t_0) = \mathbf{0}, \ldots, E^{(l)}(t_0) = \mathbf{0}$ and the derivates of evolute of order $l+1$ and $l+2$ are linearly independent.*

Proof. A vertex $P(t_0)$ of order exactly l fulfills $k^{(1)}(t_0) = \mathbf{0}, \ldots, k^{(l)}(t_0) = \mathbf{0}$ and $k^{(l+1)}(t_0) \neq \mathbf{0}$. The derivate of order $m, 1 \leq m \leq l$ of the evolute of this curve has the following expression, see Theorem 2:

$$E^{(m)} = \{\left((m-1)k^m k^{(m-1)} + \alpha_l(k^{(0)},\ldots,k^{(m-2)})\right)\mathbf{t} + \left(k^{m-1}k^{(m)} + \beta_n(k^{(0)},\ldots,k^{(m-1)})\right)\mathbf{n}\}\frac{1}{k^{m+1}}.$$

It is obvious that in this expression there are only derivates of curvature of maximal order l, so all the derivates of the evolute vanishes till order l. Compute the derivates of evolute of order $l+1$ and $l+2$:

$$E^{(l+1)} = \{\left(lk^{l+1}k^{(l)} + \alpha_{l+1}(k^{(0)},\ldots,k^{(l-1)})\right)\mathbf{t} + \left(k^l k^{(l+1)} + \beta_{l+1}(k^{(0)},\ldots,k^{(l)})\right)\mathbf{n}\}\frac{1}{k^{l+2}}.$$

$$E^{(l+2)} = \{\left((l+1)k^{l+2}k^{(l+1)} + \alpha_{l+2}(k^{(0)}, \ldots, k^{(l)})\right)\mathbf{t} + \left(k^{l+1}k^{(l+2)} + \beta_{l+2}(k^{(0)}, \ldots, k^{(l+1)})\right)\mathbf{n}\}\frac{1}{k^{l+3}}.$$

If we consider a vertex of order exactly l we get:

$$E^{(l+1)}(t_0) = \{k^l k^{(l+1)}\mathbf{n}\}\frac{1}{k^{l+2}}$$

$$E^{(l+2)}(t_0) = \{(l+1)k^{l+2}k^{(l+1)}\mathbf{t} + k^{l+1}k^{(l+2)}\mathbf{n}\}\frac{1}{k^{l+3}}.$$

The last two equations show that these derivates are linearly independent. □

As in the Euclidean case, see [4], Lemma 9.2, the evolute of a curve in the pseudo-Euclidean plane has a cusp at the point corresponding to an ordinary vertex of the base curve, see [5]. The next theorem classifies singularities of arbitrary order.

Theorem 5 (Classification of singularities of evolutes)**.** *Let $P(t)$ be a curve and $E(t)$ be the evolute of this curve. The singular points of evolute correspond to the vertices of the curve and it holds that if $P(t_0)$ is a vertex of an exact order l then the corresponding point $E(t_0)$ of the evolute is a singular point of the same order.*

In addition, the corresponding point $E(t_0)$ of the evolute is a cusp of the first kind if order l of the vertex $P(t_0)$ is odd, or it is an insignificantly singular point if order l of the vertex $P(t_0)$ is even.

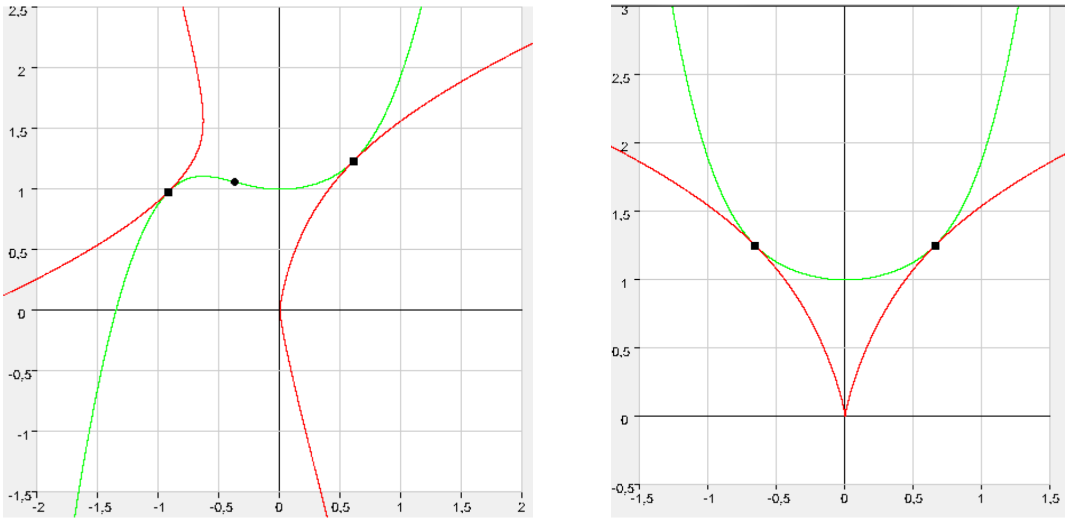
Proof. The assertion about the order of singularities of evolute is clear from Theorem 4. Let us prove assertions about the kind of singularities of the evolute.

Consider a vertex $P(t_0)$ of order exactly l of a curve $P(t)$ while l is an odd number. The previous theorem shows that $l+1$ is the lowest number satisfying $E^{(l+1)} \neq \mathbf{0}$. Because $l+1$ is even, the corresponding point of the evolute is a cusp, see Lemma 3. It also holds that $E^{(l+2)} \neq \mathbf{0}$ and $E^{(l+1)}$ and $E^{(l+2)}$ are linearly independent so the considered point of the evolute is a cusp of a first kind because $l+2$ is again odd.

Consider a vertex $P(t_0)$ of order exactly l of a curve $P(t)$ while l is an even number. The previous theorem shows that $l+1$ is the lowest number satisfying $E^{(l+1)} \neq \mathbf{0}$. Because $l+1$ is odd, the corresponding point of the evolute is an insignificantly singular point, see Lemma 3. □

Example 1. *Let us consider the curve $P(t) = (\sinh t, t^5 + \cosh t)$. By some direct computation, it can be shown that the point $P(0) = (0,1)$ of this curve is a vertex of order exactly 2. So according to the Theorem 5, the corresponding point of the evolute $E(0) = (0,0)$ is an insignificantly singular point of the evolute because the order of that singular point is even.*

Example 2. *Let us consider the curve $P(t) = (\sinh t, t^6 + \cosh t)$. It can be shown that the point $P(0) = (0,1)$ of this curve is a vertex of order exactly 3. Again according to the Theorem 5 the corresponding point of the evolute $E(0) = (0,0)$ is a cusp of the first kind because the order of that singular point is odd.*



(a) (b)

Fig. 1. (a) The evolute of the curve $P(t) = (\sinh t, t^5 + \cosh t)$, the point $E(0)$ of the evolute is an insignificantly singular point, squares indicate light-like and the dot an iflexion point of the curve.

(b) The evolute of the curve $P(t) = (\sinh t, t^6 + \cosh t)$, the point $E(0)$ of the evolute is a cusp of the first kind, squares indicate light-like points of the curve.

References

[1] BAKUROVÁ, V. On Osculating Pseudo-circles of Curves in the Pseudo-Euclidean Plane In *Proceedings of Symposium on Computer Geometry SCG'2011.* Vol. 20. Bratislava : Slovenská technická univerzita. 2011.

[2] BOŽEK, M. On Geometry of Differentiable Curves in the pseudo-Euclidean Plane In *Proceedings of Symposium on Computer Geometry SCG'2011.* Vol. 20. Bratislava : Slovenská technická univerzita. 2011.

[3] BOŽEK, M., FOLTÁN, G. On Singularities of Arbitrary Order of Pedat Curves In *Proceedings of Symposium on Computer Geometry SCG'2012.* Submitted.

[4] GIBSON, C.G. *Geometry of Differentiable Curves: an Undergraduate Introduction.* Cambridge University Press 2001. ISBN 0 521 80453 1.

[5] SALOOM, A., TARI, F. Curves in the Minkowski plane and their contact with pseudo-circles In *Geometriae Dedicata.* Volume 159. Number 1. Pages 109-124. 2012