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ON SINGULARITIES OF PEDAL CURVES IN THE MINKOWSKI PLANE

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Abstract: We point out connection between singular points on pedal curve and inflexion points on base curve in the Minkowski plane. We formulate hypotheses about the type of singular points on pedal curve according to the position of pedal point and order of corresponding inflexion points on base curve.

Keywords: Minkowski plane, pedal curve, singular point, inflexion point

1. Introduction

The concept of pedal curve is well-known in differential geometry of the Euclidean plane, see e.g. [3], [4]. In this paper, we discuss some properties of this object and formulate hypotheses about singular points of these curves in the Minkowski plane.

Authors Božek and Foltán studied singularities of pedal curves in the Euclidean plane in paper [3]. They did not obtain a complete classification of singular points of pedal curves. Our paper does not just convert their results to the Minkowski plane. We formulate hypotheses which cover all possible cases in which singular points on pedal curve occur.

In all paper we work in the Minkowski plane. *The Minkowski plane* $\mathbf{E^{1,1}}$ is the real affine plane $\mathbf{A^2}$ furnished with a nonsingular indefinite bilinear form $\langle \mathbf{x}, \mathbf{y} \rangle$ on vectors of $\mathbf{A^2}$, called *the pseudo-scalar product*. This pseudo-scalar product can be expressed as $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 - x_2 y_2$ in a suitable basis where $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$. We deal only with such bases.

For any vector $\mathbf{x} = (x_1, x_2)$ we define its *lenght* as $|\mathbf{x}| = |\langle \mathbf{x}, \mathbf{x} \rangle|^{\frac{1}{2}}$, and its *sign* as $\operatorname{sgn} \mathbf{x} = \operatorname{sgn}\langle \mathbf{x}, \mathbf{x} \rangle$.

We can approach parametrized curves in the Minkowski plane similarly as in the Euclidean plane. Every regular curve can be expressed in a unit speed parametrization for which it holds $|\langle P'(s), P'(s) \rangle| = 1$.

Moreover, for a curve expressed in a unit speed parametrization hold the Frenet formulas $\mathbf{t}'(s) = k(s)\mathbf{n}(s)$ and $\mathbf{n}'(s) = k(s)\mathbf{t}(s)$.

We refer to [1] and [2] for more details about curves in the Minkowski plane.

2. Singular points of pedal curves

Definition 2.1. Let $P(t)$ be a regular curve in the Minkowski plane and let Q be a fixed point of this plane. For every parameter t contruct orthogonal projection $F(t)$ of the point Q to the tangent line $p(t)$ of the curve $P(t)$. We call this curve $F(t)$ pedal curve of the base curve $P(t)$ with respect to pedal point Q. The pedal curve $F(t)$ of a base curve $P(t)$ can be expressed as

$$F(t) = P(t) + \frac{\langle (Q - P(t)), P'(t) \rangle}{\langle P'(t), P'(t) \rangle} P'(t).$$

In what follows, $P(t)$ is a regular curve in the Minkowski plane with tangent line $p(t)$ at the point $P(t)$. Q is pedal point and $F(t)$ is the corresponding pedal curve with tangent line $f(t)$ at the point $F(t)$.

Theorem 2.1. Pedal curve of the base curve expressed in a unit speed parametrization has these expressions

$$F(s) = P(s) + \operatorname{sgn} P'(s)\langle (Q - P(s)), \mathbf{t}(s)\rangle \mathbf{t}(s).$$

Proof. The assertion follows directly from properties of unit speed parametrization. □

Let us remind that a singular point of a curve $P(t)$ is a point $P(t_0)$ for which it holds $P^{(1)}(t_0) = \mathbf{0}$. For the classification of singular points we use the next commonly used lemma:

Lemma 2.1. Let $P(t)$ be a curve and let $P(t_0)$ be a singular point of this curve. Let k be the smallest natural number such that $P^{(k)}(t_0) \neq \mathbf{0}$.

1. If k is odd then the point $P(t_0)$ is an insignificantly singular point of the curve and the tangent line at the singular point is given by vector $P^{(k)}(t_0) \neq \mathbf{0}$.
2. If k is even then the point $P(t_0)$ is a cusp of the curve and the tangent line at the singular point is given by vector $P^{(k)}(t_0) \neq \mathbf{0}$.
3. Let k be even and let l be the smallest natural number so that $P^{(k)}(t_0)$ and $P^{(l)}(t_0)$ are linearly independent. Then the point $P(t_0)$ is a cusp of the first kind for l odd. In this case the curve locally lies on both side of the tangent line $p_P(t_0)$. The point $P(t_0)$ is a cusp of the second kind for l even and the curve locally lies on one side of the tangent line $p_P(t_0)$.

The point $P(t_0)$ is inflexion point of a curve $P(t)$ if and only if the curvature $k(t_0)$ equals zero at the point $P(t_0)$. The next definition extends this notion to inflexion point of higher order.

Definition 2.2. Point $P(t_0)$ is inflexion point of order exactly n, $n \geq 0$ of a curve $P(t)$ if $k^{(0)}(t_0) = 0, \ldots, k^{(n)}(t_0) = 0$ and $k^{(n+1)}(t_0) \neq 0$.

The next theorem shows us how to find all singular points of pedal curve.

Theorem 2.2. The point $F(t_0)$ of pedal curve is singular if and only if one of the following situations occures:

(i) pedal point Q does not coincide with $P(t_0)$ and this point is inflexion point of the base curve

(ii) pedal point Q coincide with non-inflextion point $P(t_0)$ of the base curve

(iii) pedal point Q coincide with inflextion point $P(t_0)$ of the base curve

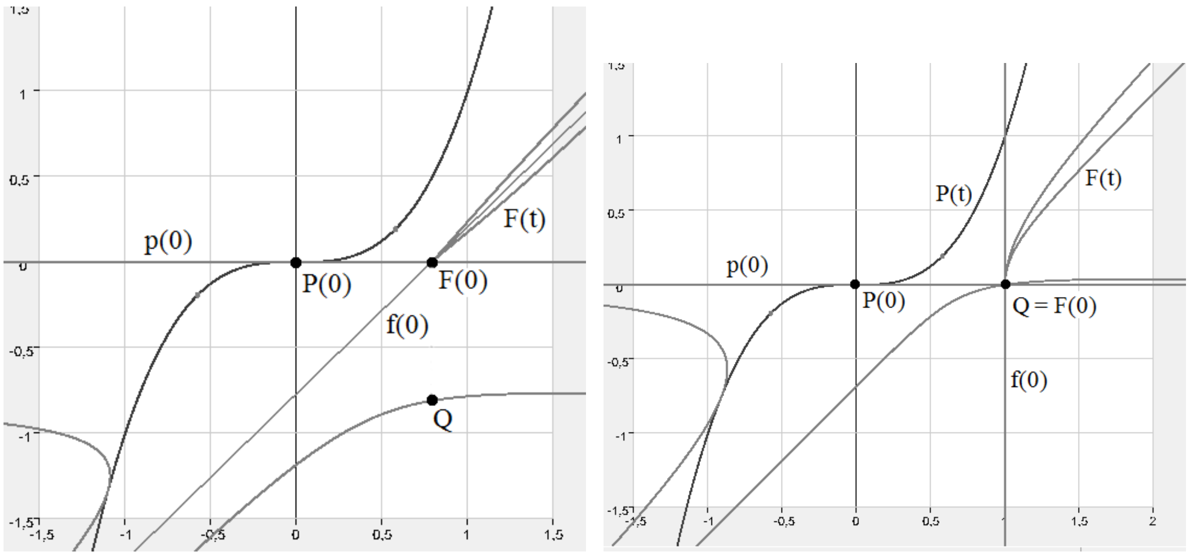
Proof. $P(t)$ is a regular curve so we can express it in an unit speed parametrization. The pedal curve can be expressed as

$$F(s) = P(s) + \operatorname{sgn} P'(s)\langle (Q - P(s)), \mathbf{t}(s)\rangle \mathbf{t}(s).$$

Using a straight-forward calculation and Frenet formulas we get

$$F'(s) = \operatorname{sgn} P'(s) k(s) \left(\langle (Q - P(s)), \mathbf{n}(s)\rangle \mathbf{t}(s) + \langle (Q - P(s)), \mathbf{t}(s)\rangle \mathbf{n}(s) \right).$$

Vectors $\mathbf{t}(s)$ and $\mathbf{n}(s)$ are linearly independent so $F'(s) = 0$ if $k(s)\langle (Q - P(s)), \mathbf{t}(s)\rangle = 0$ and $k(s)\langle (Q - P(s)), \mathbf{n}(s)\rangle = 0$. From this follows that $F'(s) = 0$ if and only if $k(s) = 0$ or $P(s) = Q$. □



(a) (b)

Fig. 1. The pedal curve of the curve $P(t) = (t, t^3)$, $F(0)$ is (a) a cusp of the first kind and (b) a cusp of the second kind.

3. Classification of singular points of pedal curves

The aim of this section is to formulate hypotheses about the type of the singular point on pedal curve according to the position of pedal point and according to order of the corresponding inflexion point on base curve. For this aim we use a special visualization tool created for this purpose.

Let us consider the first case mentioned in Theorem 2.2. We formulate the following hypothesis about singular points of pedal curve.

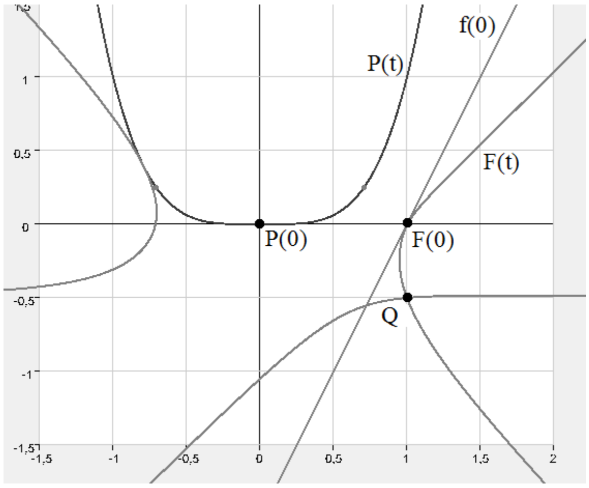
Hypothesis 3.1. Let $P(t)$ be a regular curve with inflexion point $P(t_0)$ of order exactly n. Let $F(t)$ be the corresponding pedal curve such that pedal point Q does not coincide with the inflexion point $P(t_0)$. Then $F(t_0)$ is an insignificantly singular point of pedal curve if n is odd. If n is even, the point $F(t_0)$ is a cusp of the first kind if the pedal point Q does not lie on the tangent line of the base curve at $P(t_0)$ and it is a cusp of the second kind if the pedal point Q lies on the tangent line of the base curve at $P(t_0)$.

The following figures and examples support this hypothesis.

Example: The curve $P(t) = (t, t^3)$ has one inflexion point $P(0)$ of order exactly 0 (even order). Figure 1 shows us the situation when $Q \neq P(0)$ and the corresponding point $F(0)$ of pedal curve is singular. In case a) Q does not lie on $p(0)$ so $F(0)$ is a cusp of the first kind. Really, pedal curve lies locally on both sides of the tangent line $f(0)$. In case b) Q lies on $p(0)$ so $F(0)$ is a cusp of the second kind. Really, pedal curve lies locally on one side of the tangent line $f(0)$.

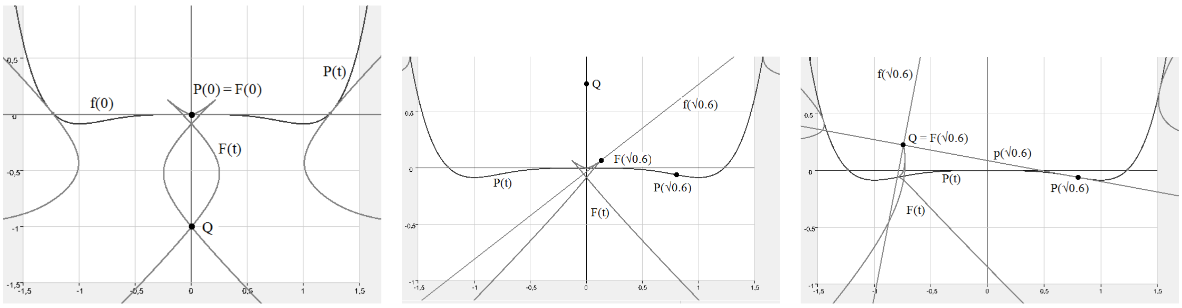
Example: The curve $P(t) = (t, t^4)$ has one inflexion point $P(0)$ of order exactly 1. Figure 2 shows us the situation when $Q \neq P(0)$ and the corresponding point $F(0)$ of pedal curve is singular. In this case $F(0)$ is an insignificantly singular point.

Example: The curve $P(t) = (t, \frac{1}{6}t^6 - \frac{1}{4}t^4)$ has exactly three inflexion points: $P(0)$ of order



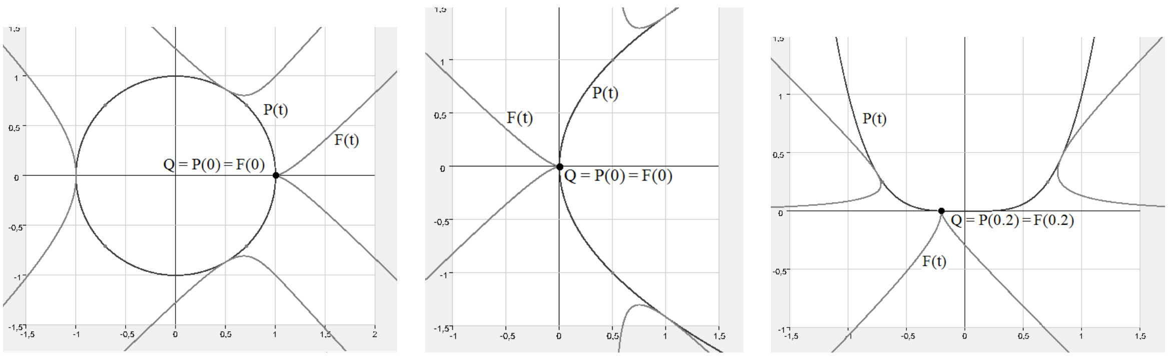
(a)

Fig. 2. The pedal curve of the curve $P(t) = (t, t^4)$, $F(0)$ is an insignificantly singular point.



(a) (b) (c)

Fig. 3. The pedal curve of the curve $P(t) = (t, \frac{1}{6}t^6 - \frac{1}{4}t^4)$, (a) $F(0)$ is an insignificantly singular point, (b) $F(\sqrt{0.6})$ is a cusp of the first kind, c) $F(\sqrt{0.6})$ is a cusp of the second kind.



(a) (b) (c)

Fig. 4. The pedal curve of (a) a Euclidean circle $P(t) = (\cos t, \sin t)$, (b) a parabola $P(t) = (t^2, 2t)$ and c) the curve $P(t) = (t, t^4)$. In all cases the singular point of the pedal curve is a cusp of the first kind.

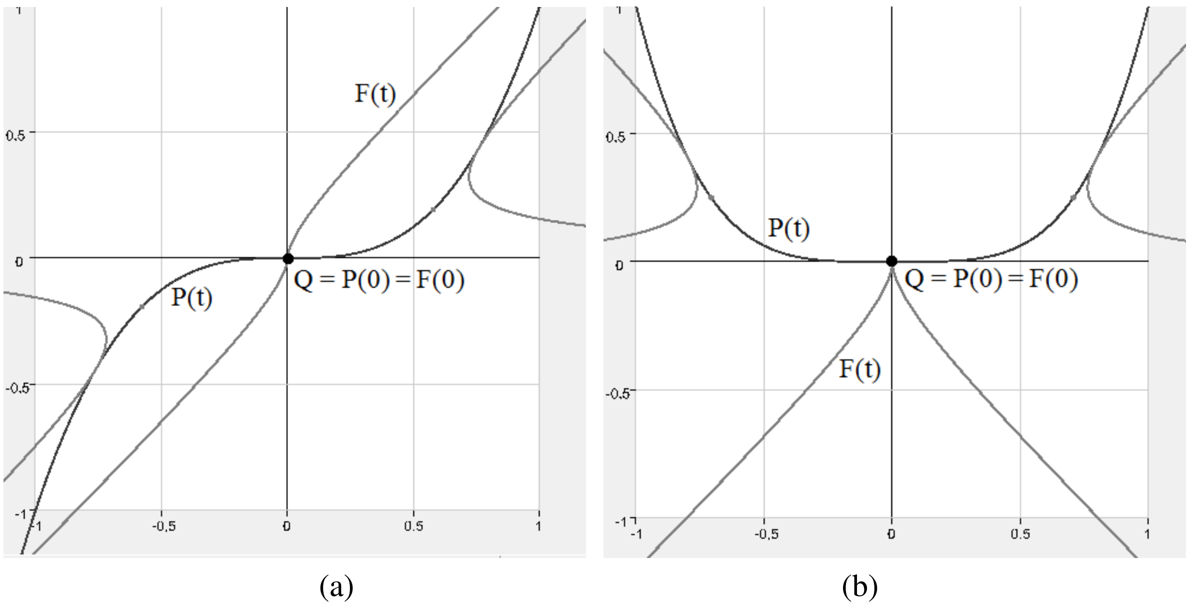


Fig. 5. The pedal curve of (a) the curve $P(t) = (t, t^3)$, $F(0)$ is an insignificantly singular point and (b) the curve $P(t) = (t, t^4)$, $F(0)$ is a cusp of the first kind.

exactly 1 and $P(\pm\sqrt{0.6})$ of order exactly 0. Figure 3 shows us the situation when Q does not coincide with inflexion point and the corresponding point of pedal curve is singular. In case a) $F(0)$ is an insignificantly singular point. In case b) Q does not lie on $p(\sqrt{0.6})$ so the singular point $F(\sqrt{0.6})$ is a cusp of the first kind. In case c) Q lies on $p(\sqrt{0.6})$ so the singular point $F(\sqrt{0.6})$ is a cusp of the second kind.

Let us consider the second case from Theorem 2.2 and formulate the following hypothesis.

Hypothesis 3.2. Let $P(t)$ be a regular curve and $F(t)$ be the corresponding pedal curve such that the pedal point Q coincide with non-inflexion point $P(t_0)$ of the base curve. Then $F(t_0)$ is a cusp of the first kind.

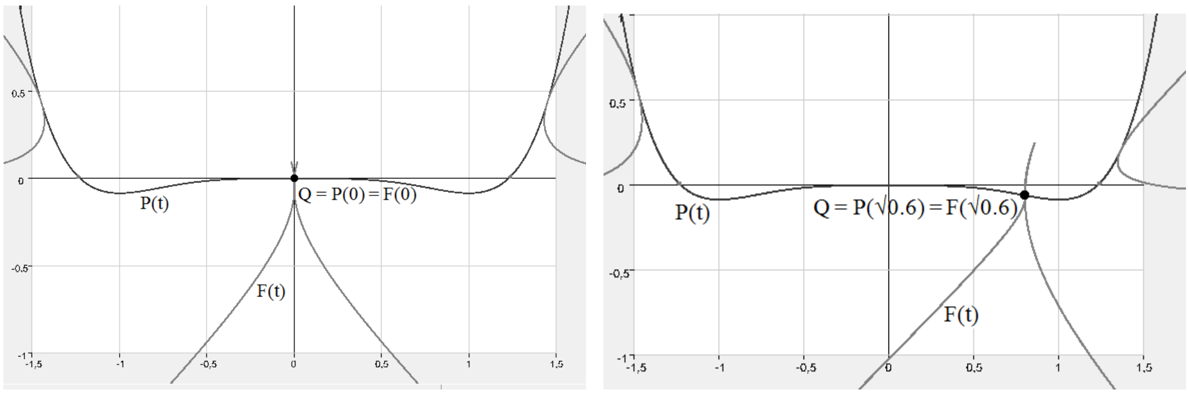
The following figure and example support this hypothesis.

Example: Let us consider a Euclidean circle $P(t) = (\cos t, \sin t)$ and the pedal point $Q = P(0)$, a parabola $P(t) = (t^2, 2t)$ and the pedal point $Q = P(0)$ and the curve $P(t) = (t, t^4)$ and the pedal point $Q = P(0.2)$. In all cases the pedal point Q coincide with a non-inflextion point of the base curve. Figure 4 shows us the described situations. It is obvious that points $F(0)$ and $F(0.2)$, respectively, are singular and are cusps of the first kind.

Let us consider the last situation from Theorem 2.2 and formulate the next hypothesis supported by the following figures and examples.

Hypothesis 3.3. Let $P(t)$ be a regular curve with inflexion point $P(t_0)$ of order exactly n. Let $F(t)$ be the corresponding pedal curve such that pedal point Q coincides with the inflexion point $P(t_0)$. Then $F(t_0)$ is an insignificantly singular point of pedal curve if n is even or is a cusp of the first kind if n is odd.

Example: The curve $P(t) = (t, t^3)$ has one inflexion point $P(0)$ of order exactly 0. The curve $P(t) = (t, t^4)$ has the same inflexion point $P(0)$ but of order exactly 1. Figure 5 shows us the



(a) (b)

Fig. 6. The pedal curve of the curve $P(t) = (t, \frac{1}{6}t^6 - \frac{1}{4}t^4)$, (a) $F(0)$ is a cusp of the first kind and (b) $F(\sqrt{0.6})$ is an insignificantly singular point.

situation when $Q = P(0)$ in both cases and the point $F(0)$ is singular. In case a) is $F(0)$ an insignificantly singular point while in case b) $F(0)$ is a cusp of the first kind.

Example: The curve $P(t) = (t, \frac{1}{6}t^6 - \frac{1}{4}t^4)$ has three inflexion points $P(0)$ of order exactly 1 and $P(\pm\sqrt{0.6})$ of order exactly 0. Figure 6 shows us the situation when Q coincides with the inflexion point of the base curve. In case a) $Q = P(0)$ and $F(0)$ is a cusp of the first kind. In case b) $Q = P(\sqrt{0.6})$ and $F(\sqrt{0.6})$ is an insignificantly singular point.

Our future work will be to proove the above formulated hypotheses.

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