Notes on Evolutes in the Minkowski Plane

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Abstract
In this paper, we discuss some properties of evolutes of curves in the Minkowski plane.

Keywords: Minkowski plane, evolute, convex curve

Abstrakt
Tento článok sa venuje vybraným vlastnostiam evolút kriviek ležiacich v Minkowského rovine.

Kľúčové slová: Minkowského rovina, evolúta, konvexná krivka

1 Introduction

The concept of evolute is well-known in differential geometry of the Euclidean plane. In [2] we have shown some properties of evolute in the Minkowski plane which correspond to the Euclidean case.

Authors Saloom and Tari studied smooth and regular curves in the Minkowski plane related by their contact with pseudo-circles in paper [5]. They arrived at the concept of evolute in quite a natural way and obtained interesting results about collocation of points of evolute with respect to points of basic curve. Some of the mentioned properties have no analogy in the Euclidean plane. After slight modifications, the mentioned properties can be formulated as follows:

Theorem 1.1. ([5], Prop. 3.3) Let $P(t)$ be a connected space-like or time-like curve. Then $P(t)$ does not intersect its evolute.

Theorem 1.2. ([5], Prop. 4.1) Let $P(t)$ be a curve without inflection points. Then

(i) the light-like points of $P(t)$ are isolated points,

(ii) the caustic of $P(t)$ is a regular curve at a light-like point of $P(t)$ and has ordinary tangency with $P(t)$ at such point. Furthermore, $P(t)$ and its caustic lie locally on opposite sides of their common tangent line at the light-like point.

Theorem 1.3. ([5], Th. 4.3) Let $P(t)$ be an oval in the Minkowski plane. Then,

(i) $P(t)$ has exactly four light-like points,

(ii) the caustic of $P(t)$ is a closed curve which lies in the complement of the interior of $P(t)$,

(iii) the evolute of each space-like and time-like component of $P(t)$ has at least one singular point.

Saloom and Tari applied tools from theory of singularities to obtain geometric information about evolutes in the Minkowski plane. We consider such approach being a rather complicated one. Moreover, their proof of Theorem 1.3 (ii) is not sufficient because they did not disprove a case when the evolute of a space-like or time-like component of an oval possibly intersects other such component of the oval.

The goal of our paper is to re-prove Theorems 1.2 and 1.3 using only elementary tools of classical differential geometry. Particularly, we prefer to write 'augmented evolute' instead of 'caustic' from [5] when speaking about evolute defined also at light-like points.

2 Preliminaries

The Minkowski plane $\mathbf{E^{1,1}}$ is the real affine plane $\mathbf{A^2}$ furnished with a nonsingular indefinite bilinear form $\langle \mathbf{x}, \mathbf{y} \rangle$ on vectors of $\mathbf{A^2}$, called *the pseudo-scalar product*. This pseudo-scalar product can be expressed as $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 - x_2 y_2$ in a suitable basis where $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$. In our paper we deal only with such bases.

For any vector $\mathbf{x} = (x_1, x_2)$ we define its *lenght* as $|\mathbf{x}| = |\langle \mathbf{x}, \mathbf{x} \rangle|^{\frac{1}{2}}$, and its *sign* as $\operatorname{sgn} \mathbf{x} = \operatorname{sgn} \langle \mathbf{x}, \mathbf{x} \rangle$. We say that $\mathbf{x}$ is *a space-like vector* or *a time-like vector* or *a light-like vector* if $\operatorname{sgn} \mathbf{x} = 1$ or $\operatorname{sgn} \mathbf{x} = -1$ or $\operatorname{sgn} \mathbf{x} = 0$, respectively.

We say that vectors $\mathbf{x}$ and $\mathbf{y}$ are *perpendicular* if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. Perpendicularity of these vectors is denoted by $\mathbf{x} \perp \mathbf{y}$.

There are two useful operators on vectors in the Minkowski plane: *The symmetry operator* $\mathbf{x} \to \mathcal{S}\mathbf{x}$ assigning the vector $\mathcal{S}\mathbf{x} = (x_2, x_1)$ to vector $\mathbf{x}$, and *the perpendicularity operator* $\mathbf{x} \to \mathbf{x}^\perp$ which assigns the vector $\mathbf{x}^\perp = \operatorname{sgn} \mathbf{x} \mathcal{S}\mathbf{x}$ to vector $\mathbf{x}$. Both vectors $\mathcal{S}\mathbf{x}$ and $\mathbf{x}^\perp$ are perpendicular to $\mathbf{x}$. Besides this, the basis $(\mathbf{x}, \mathbf{x}^\perp)$ is positively oriented for every non-light-like vector $\mathbf{x}$. Apparently, the symmetry operator $\mathcal{S}$ is linear.

Fig. 1. Vectors $\mathcal{S}\mathbf{x}$ and $\mathbf{x}^\perp$ assigned to vector $\mathbf{x} = (-1, 2)$. Both vectors $\mathcal{S}\mathbf{x}$ and $\mathbf{x}^\perp$ are perpendicular to $\mathbf{x}$, the basis $(\mathbf{x}, \mathbf{x}^\perp)$ is positively oriented

From the coordinate expression of the pseudo-scalar product and from the previous definitions we come to the following properties of vectors in the Minkowski plane.

$$\langle \mathbf{x}, \mathbf{y} \rangle = \det(\mathbf{x}, \mathcal{S}\mathbf{y}) \tag{1}$$
$$\det(\mathbf{x}, \mathbf{y}) = -\det(\mathcal{S}\mathbf{x}, \mathcal{S}\mathbf{y}) \tag{2}$$
$$\det(\mathbf{x}, \mathbf{y})\mathbf{z} = \det(\mathbf{x}, \mathbf{z})\mathbf{y} + \det(\mathbf{z}, \mathbf{y})\mathbf{x} \tag{3}$$
$$\mathbf{x} \text{ is light-like } \iff \mathcal{S}\mathbf{x} = \delta\mathbf{x}, \text{ where } \delta \in \{-1, 1\} \tag{4}$$

We can approach parametrized curves in the Minkowski plane similarly as in the Euclidean plane. According to the type of the tangent vector at a point of a curve, the point is said to be a *space-like* or a *time-like* or a *light-like point of the curve*.

At every regular and not light-like point $P(t)$ of a curve we have *the oriented Frenet frame* consisting of vectors

$$\mathbf{t}(t) = \frac{P'(t)}{|P'(t))|}, \qquad \mathbf{n}(t) = \mathbf{t}(t)^{\perp}$$

and *the (oriented) curvature*[1]

$$k(t) = \frac{\det(P'(t), P''(t))}{|\langle P'(t), P'(t)\rangle|^{\frac{3}{2}}} = -\frac{\langle P''(t), \mathcal{S}P'(t)\rangle}{|\langle P'(t), P'(t)\rangle|^{\frac{3}{2}}}.$$

In what follows, we often simplify formulas by omitting the variable (or its particular value) in parametrizations of curves and in their derivatives.

3 Evolutes in the Minkowski plane

The aim of this section is to describe some properties of evolute in the Minkowski plane. As in the Euclidean plane, evolute of a curve in the Minkowski plane is the locus of centers of all its osculating (pseudo-)circles. Let us note that pseudo-circles in the Minkowski plane are equilateral hyperbolas with light-like asymptotes from the Euclidean point of view. In the formula for centre of osculating pseudo-circle of a curve in the Minkowski plane (see [1] or [5]), there appears a sign-change comparing to the Euclidean case. This is reflected in the following definition.

Definition 3.1. *Evolute* of a curve $P(t)$, with its inflexion and light-like points removed, is defined as

$$E(t) = P(t) - \frac{1}{k(t)}\mathbf{n}(t).$$

Using the definition of the normal vector $\mathbf{n}(t)$ and of the curvature $k(t)$ we get easily a handy computational formula

$$E(t) = P(t) - \frac{\langle P'(t), P'(t)\rangle}{\det(P'(t), P''(t))}\mathcal{S}P'(t) \tag{5}$$

[1] Saloom and Tari in [5], Sec. 3 also present a formula for oriented curvature but with incorrect sign which is equal to $\operatorname{sgn} P'(t)$.

We emphasize that the evolute $E(t)$ is not defined at inflexion and light-like points of the curve $P(t)$. Nevertheless, we wish to extend the definition of evolute also for light-like points of the basic curve. We will use *an equi-affine parametrization* of the basic curve for that sake. Such parametrization is defined by the condition

$$\det(P'(t), P''(t)) = 1 \tag{6}$$

and it exists for any curve not containing inflexion points, see e.g. [4], §10 for more details.

From identities (1), (2) and (6) it immediately follows

$$\langle \mathcal{S}P'(t), P''(t)\rangle = \det(\mathcal{S}P'(t), \mathcal{S}P''(t)) = -1 \tag{7}$$

for every equi-affine parametrization $P(t)$.

Lemma 3.1. Let $P(t)$ be a curve without inflexion points expressed in an equi-affine parametrization. Then

(a) $E(t) = P(t) - \langle P'(t), P'(t)\rangle \mathcal{S}P'(t)$

(b) $E'(t) = -3\langle P'(t), P''(t)\rangle \mathcal{S}P'(t)$

(c) $E''(t) = -3\langle P'(t), P''(t)\rangle \mathcal{S}P''(t) - 3\langle P'(t), P''(t)\rangle' \mathcal{S}P'(t)$

Proof. **(a)** The assertion is a simple consequence of formulas (5) and (6).

(b) Using a straight-forward calculation we get

$$\begin{aligned} E' &= P' - \langle P', P'\rangle' \mathcal{S}P' - \langle P', P'\rangle \mathcal{S}P'' \\ &= P' - 2\langle P', P''\rangle \mathcal{S}P' - \langle P', P'\rangle \mathcal{S}P'' \end{aligned}$$

Further, equalities (1), (3), (2) and (7) imply

$$\begin{aligned} E' &= P' - 2\langle P', P''\rangle \mathcal{S}P' - \det(P', \mathcal{S}P')\mathcal{S}P'' \\ &= P' - 3\langle P', P''\rangle \mathcal{S}P' - \langle \mathcal{S}P'', P'\rangle P' \\ &= P' - 3\langle P', P''\rangle \mathcal{S}P' - \det(P', P'')P' = -3\langle P', P''\rangle \mathcal{S}P' \end{aligned}$$

(c) Now we can easily compute the second derivative of this evolute

$$E'' = -3\langle P', P''\rangle \mathcal{S}P'' - 3\langle P', P''\rangle' \mathcal{S}P'$$

□

The right-hand side of the equality in Lemma 3.1 a) is defined also at light-like points, the value is $P(t)$, see Lemma 3.2 below. We call the corresponding curve *the augmented evolute* of the considered curve. By abuse of notation, we denote the augmented evolute by $E(t)$, again. Apparently, this curve is differentiable also at light-like points.

The next lemma provides values of the augmented evolute and its first two derivatives at light-like points, considering a basic curve in an equi-affine parametrization.

Lemma 3.2. Let $P(t_0)$ be a light-like point of a curve expressed in an equi-affine parametrization. Then

(a) $E(t_0) = P(t_0)$

(b) $E'(t_0) = 3P'(t_0)$

(c) $E''(t_0) = 3\delta\mathcal{S}P''(t_0) - 3\langle P'(t_0), P''(t_0)\rangle'\mathcal{S}P'(t_0), \qquad \delta \in \{-1, 1\}$

(d) $\det(P'(t_0), E''(t_0)) = -3$

Proof. **(a)** From the expression of evolute in Lemma 3.1, the equality $P(t_0) = E(t_0)$ obviously holds at light-like points.

(b) Let us consider the first derivative of evolute from Lemma 3.1 at light-like point. Using the property (4) of light-like points and formulas (1) and (7) we get

$$E' = -3\langle P', P''\rangle\mathcal{S}P' = -3\det(P', \mathcal{S}P'')\mathcal{S}P' = -3\det(\delta\mathcal{S}P', \mathcal{S}P'')\delta P' = 3P'$$

(c) In virtue of Lemma 3.1 c) and formulas (7) we have

$$E'' = -3\langle\delta\mathcal{S}P', P''\rangle\mathcal{S}P'' - 3\langle P', P''\rangle'\mathcal{S}P' = 3\delta\mathcal{S}P'' - 3\langle P', P''\rangle'\mathcal{S}P'$$

(d) Calculating $\det(P'(t_0), E''(t_0))$ we make use of the previous lemma and of the relations (4) and (7):

$$\begin{aligned}\det(P', E'') &= \det(P', 3\delta\mathcal{S}P'' - 3\langle P', P''\rangle'\mathcal{S}P') \\ &= 3\delta\det(\delta\mathcal{S}P', \mathcal{S}P'') - 3\langle P', P''\rangle'\det(\delta\mathcal{S}P', \mathcal{S}P') = -3\end{aligned}$$

□

Theorem 3.1. The augmented evolute $E(t)$ of a curve $P(t)$ without inflexion points is a regular curve at light-like points with the value $E(t_0) = P(t_0)$ at such point. The basic curve and its augmented evolute have common tangent line at light-like point. Moreover, the augmented evolute and the curve lie locally in opposite half-planes considering the common tangent line at light-like point.

Proof. Just, the assertions of the foregoing lemma were reformulated geometrically (after choosing an equi-affine parametrization of the basic curve). □

Note 3.1. The last theorem is equivalent to the second part of Theorem 1.2. What is different is that we proved the described properties within classical differential geometry.
The first assertion of Theorem 1.2 can be proven using elementary methods, as well. Indeed, let us assume that we have given a sequence of light-like points $P(t_n)$ and that $t_0 = \lim t_n$, $t_0 \neq t_n$ for all n. Obviously, the point $P(t_0)$ is light-like, too. We apply the Rolle´s Theorem to the function $\langle P'(t), P'(t)\rangle$ on the closed interval with end-points t_0, t_n. Hence, there is a point u_n in that interval such that $\langle P'(u_n), P''(u_n)\rangle = 0$. Therefore $\langle P'(t_0), P''(t_0)\rangle = 0$. As $P(t_0)$ is light-like, $\mathcal{S}P'(t_0) = \delta P'(t_0)$, $\delta \in \{-1, 1\}$. This together with (2) yields that

$$\det(P'(t_0), P''(t_0)) = -\langle\mathcal{S}P'(t_0), P''(t_0)\rangle = \delta\langle P'(t_0), P''(t_0)\rangle = 0$$

This is a contradiction to the assumption that light-like points are non-inflexion ones.

4 Evolutes of convex curves and ovals

Under *a convex curve* we mean a regular planar curve that lies on one side of each its tangent line (in supporting half-plane); c.f. [3], Chapter 1.7. At a non-inflexion point, the supporting half-plane of a convex curve is determined by the second derivative vector of the curve parametrization. Clearly, supporting half-planes contain the convex hull of our curve.

Theorem 4.1. The augmented evolute of a (closed or non-closed) convex curve without inflexion points in the Minkowski plane does not intersect interior of the convex closure of such curve.

Proof. The assertion holds trivially at light-like points of the considered curve. Let $P(t)$, $t \in I$ be an equi-affine parametrization of a non-light-like component of such curve. With respect to Lemma 3.1 b) it suffices to prove that the vectors $E(t) - P(t) = -\langle P'(t), P'(t)\rangle \mathcal{S}P'(t)$ and $P''(t)$ are pointed at different sides of the tangent line at $P(t)$. Therefore, we have to prove that $\det(P'(t), \langle P'(t), P'(t)\rangle \mathcal{S}P'(t)) \det(P'(t), P''(t)) > 0$ everywhere. However the inequality holds trivially because of (1) and (6). $\square$

Note 4.1. As a straight-forward consequence of Theorem 4.1 we get that augmented evolute of a closed convex curve without inflexion points does not intersect the interior of such curve. (The interior of a simple closed planar curve is meant in sense of Jordan's Theorem.) Because ovals were understood as simple closed curves without inflexion points in [5], and because such curves are necessarily convex (see [3], Chap. 5-7, Prop. 1) we proved the second assertion of Theorem 1.3.

Note 4.2. The first assertion of Theorem 1.3 can be slightly generalized to the case of convex curves: Every convex curve without inflexion points contains at most four light-like points; if the curve is closed, it contains exactly four such points. The reason for the first statement is that every convex curve has at most two tangent lines with given direction, and any tangent line of a convex curve without inflexion points meets the curve in exactly one point. The second statement is the original one from [5] because such curve is an oval.

Note 4.3. The validity domain of the third assertion of Theorem 1.3 can be extended without changing the proof presented in [5]: Between any two adjacent light-like points of a curve not containing inflexion points, there lies at least one vertex. Then, in virtue of [5], Prop. 3.2 or [1], Prop. 3, the corresponding part of the evolute contains a singular point.

5 Examples

We will illustrate achieved results on evolutes in the Minkowski plane through representatives of three kinds of basic curves: ovals (ellipse), convex curves that are not ovals (parabola) and nonconvex curves (Archimedean spiral). In all cases we point up, both analytically as well as graphically, light-like points of basic curves and the tangent lines at those points which separate (at most) locally the curve from its evolute.

Example: Let us consider a Euclidean ellipse with parametrization $P(t) = (a \sin t, b \cos t)$, $t \in [0, 2\pi]$, see fig. 2, left. Points of this curve are light-like if and only if $a^2 \sin^2 t - b^2 \cos^2 t = 0$. The augmented evolute of the considered ellipse is

$$E(t) = (a \sin t, b \cos t) - \frac{a^2 \sin^2 t - b^2 \cos^2 t}{ab}(b \cos t, -a \sin t).$$

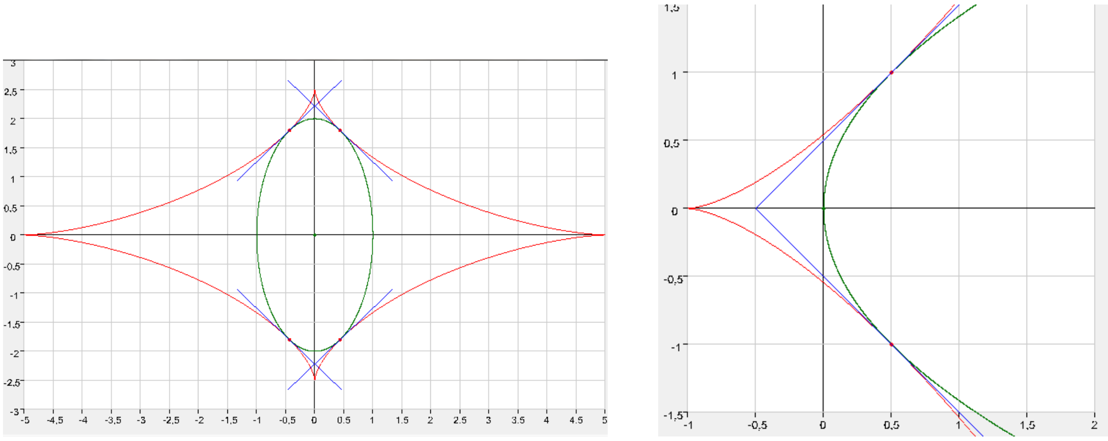

Fig. 2. Left: the augmented evolute of the Euclidean ellipse.
Right: the augmented evolute of a Euclidean parabola

Example: Let us consider a Euclidean parabola with parametrization $P(t) = (t^2/2p, t)$, where $t \in (-\infty, \infty)$, see fig. 2, right. The light-like points are described by equation $|t| = p$. The augmented evolute is

$$E(t) = \left(\frac{t^2}{2p}, t\right) + \frac{t^2 - p^2}{t}\left(1, \frac{t}{p}\right).$$

Example: We have the Archimedean spiral $P(t) = (t\cos t, t\sin t), t \in (0, \infty)$, see fig. 3. Some light-like points of the spiral such as $P(0.403) = (0.37; 0.158)$, $P(1.404) = (0.233; 0.138)$, $P(2.71) = (-0.246; 0.113), \ldots$ are described by equation $\tan 2t = \frac{1-t^2}{2t}$. The augmented evolute is

$$E(t) = (t\cos t, t\sin t) - \frac{(1-t^2)\cos 2t - 2t\sin 2t}{2+t^2}(\sin t + t\cos t, \cos t - t\sin t).$$

Let us note that that spiral is a non-convex curve and that the corresponding evolute does intersect the basic curve.

Summary

Our paper discussed some properties of evolutes in the Mikowski plane. It was motivated by the paper [5]. The goal was to re-prove some theorems from that article using only elementary methods of classical differential geometry. We consider our approach to be more adequate to the discussed topics, in contrast to the theory of singularities which was used by authors of aforementioned article. On the other side we must admit that the scope of their paper is much wider than of our one.

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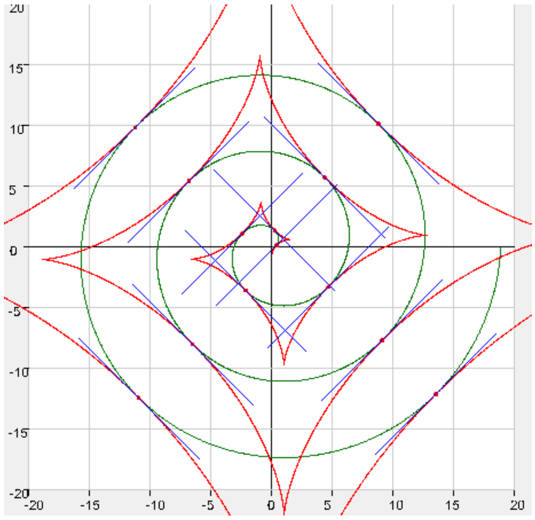

Fig. 3. The augmented evolute of the Archimedean spiral

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