Tensor-product Bézier patches

1. Represent the graph of the polynomial function

$$f(u,v) = u^2 + 4uv + u - 1$$

as a tensor-product Bézier patch (TPBP) $\mathcal{S}$ over the domain $\mathcal{D} := \langle 0,1\rangle \times \langle 0,1\rangle$.
Represent the border isocurves of $\mathcal{S}$ as

   (a) polynomial curves, i.e. in their monomial form,
   (b) Bézier curves, i.e. calculate the coordinates of their control vertices.

Calculate the coordinates of the image of $(1/2, 1/2) \in \mathcal{D}$

   (a) directly, i.e. via substitution,
   (b) using the de Casteljau algorithm,
   (c) using the bilinear calculation method.

Represent the graph of $f(u,v)$ as TPBP over $\mathcal{D}$ via the polar form of $f(u,v)$.

2. Assume a bilinear patch $\mathcal{S}$ with control vertices

$$p_{00} = (0,0,2),\ p_{10} = (1,0,-1),\ p_{01} = (0,1,1),\ p_{11} = (1,1,0),$$

where $\mathcal{S}$ is an image of the domain $\mathcal{D} := \langle 0,1\rangle \times \langle 0,1\rangle$.
Find the parametric and analytical equations of the tangent plane of $\mathcal{S}$ at the image of $(1/2, 1/2) \in \mathcal{D}$.
Assume a quadratic Bézier curve $\mathcal{Q} \subset \mathcal{D}$ with control vertices

$$q_0 = (0,0),\ q_1 = (0,1),\ q_2 = (1,0).$$

Represent the image of $\mathcal{Q}$ lying on $\mathcal{S}$ as a Bézier curve, i.e. calculate the coordinates of its control vertices.

RNDr. Martina Bátorová, PhD.
martina.batorova@fmph.uniba.sk

C^k-continuous composition of tensor-product Bézier patches

3. Assume a patch $\mathcal{S}$ created via composition of two TPBP $\mathcal{S}_1$ and $\mathcal{S}_2$ with control vertices as in fig. 1; the domains of both $\mathcal{S}_1, \mathcal{S}_2$ are unit squares.

   Assume a curve $\mathcal{C}_1 \subset \mathcal{S}$ passing through the middle of both the patches, i.e. the u-curve for $v = 1/2$ (see fig. 2).

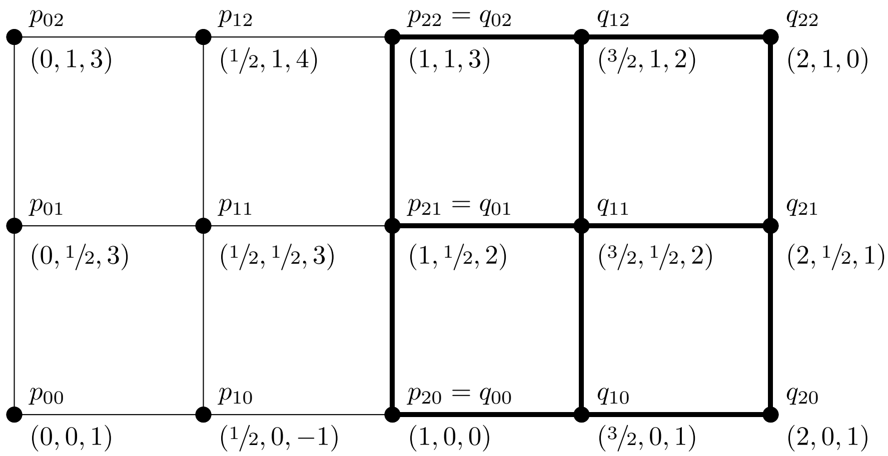

Fig. 1: Coordinates of the patches $\mathcal{S}_1$ (left) and $\mathcal{S}_2$ (right).

   Determine the order k of the C^k-continuity of $\mathcal{S}$ along the common isocurve.

   Determine the order k of the C^k-continuity of the curve $\mathcal{C}_1$.

   If necessary, change the coordinates of p_{11} resp. p_{10} so that $\mathcal{C}_1$ is C^1-continuous.

4. Assume the patches $\mathcal{S}_{11}$ and $\mathcal{S}_{10}$ from previous example, with $\mathcal{S}_{11}$ resp. $\mathcal{S}_{10}$ being the patch $\mathcal{S}$ after the change of coordinates of p_{11} resp. p_{10}.

   Determine the order k of the C^k-continuity of the curves $\mathcal{C}_2$ and $\mathcal{C}_3$, where $\mathcal{C}_2$ is the image of the diagonal of the domain and $\mathcal{C}_3$ is the u-curve corresponding to $v = 1/4$ (see fig. 2).

   Remark: Determine the order k of the C^k-continuity of the curves $\mathcal{C}_2, \mathcal{C}_3$ in both the patches $\mathcal{S}_{11}, \mathcal{S}_{10}$.

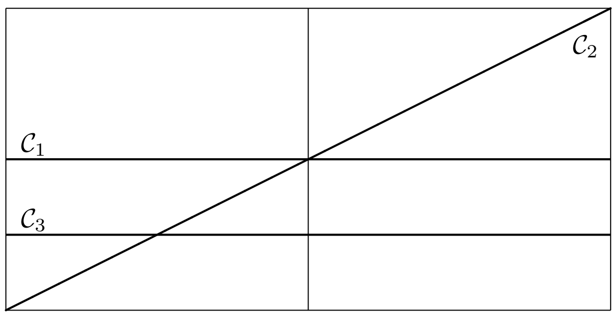

Fig. 2: Curves $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ in the domain $\mathcal{D}$ of the patch $\mathcal{S}_{11}$ resp. $\mathcal{S}_{10}$.

RNDr. Martina Bátorová, PhD.
martina.batorova@fmph.uniba.sk

Triangular Bézier patches

5. Assume three non-collinear points $p_0, p_1, p_2 \in \mathbb{A}^2(\mathbb{R})$ and use them to define a barycentric coordinate system $\mathcal{B}$.

   Determine the barycentric representation of

   (a) a point $A \in \triangle p_0 p_1 p_2$ and the points $p_i, i = 0, 1, 2$,

   (b) a line ℓ that is parallel to any side of $\triangle p_0 p_1 p_2$ and the sides of $\triangle p_0 p_1 p_2$.

6. Represent the graph of a polynomial function

   $$F(u, v) = 3 + 6u - 2v - u^2 + 4uv + 2v^2$$

   as a triangular Bézier patch $b^\triangle \subset \mathbb{R}^3$ defined over a domain

   $$\mathcal{D}^\triangle := \{(u, v) \mid u, v \geq 0,\ u + v \leq 1\} \subset \mathbb{R}^2,$$

   i.e. determine the coordinates of its control vertices. Proceed using

   (a) a direct substitution into the formula

   $$b^\triangle(t) = \sum_{i+j+k=n} B^n_{ijk}(t) \cdot p_{ijk},$$

   with $n \in \mathbb{N}$ being the degree of the patch $b^\triangle(t)$.
   Remark: $B^n_{ijk}(t) = \binom{n}{ijk} t_0^i\, t_1^j\, t_2^k$ are Bernstein polynomials of degree n in two variables and the domain $\mathcal{D}^\triangle$ is a barycentric coordinate system s.t. any $t \in \mathbb{R}^2$ can be expressed as $t = (t_0, t_1, t_2)$ with $t_0 + t_1 + t_2 = 1$.

   (b) a subdivision of the domain $\mathcal{D}^\triangle$, i.e. using the properties of the polynomials $B^n_{ijk}(t)$,

   (c) the polar form $\phi^\triangle(\bar{t})$ of the polynomial $F(u, v)$.

   Determine the border curves of such a patch $b^\triangle(t)$ and verify your results using $F(u, v)$.

   Calculate the coordinates of the image P of $p = (1/4, 1/2) \in \mathcal{D}^\triangle$ using

   (a) a straightforward substitution,

   (b) the de Casteljau algorithm,

   (c) the polar form of the polynomial $F(u, v)$

   Determine the equation of the tangent plane at P.

   Subdivide the patch at P and represent the subpatches as triangular Bézier patches.

C^k-continuous composition of triangular and tensor-product Bézier patches

7. Assume two quadratic triangular Bézier patches $b^\triangle$ resp. $c^\triangle$ defined over domains $\mathcal{D}_{b^\triangle} := \triangle p_0 p_1 p_2$ resp. $\mathcal{D}_{c^\triangle} := \triangle p'_0 p_1 p_2$, see fig. 3.

   Calculate the coordinates of the remaining control vertices of the patch $c^\triangle$ so that the patches $b^\triangle$ and $c^\triangle$ are joined in a C^1- resp. C^2-continuous manner along the common border.

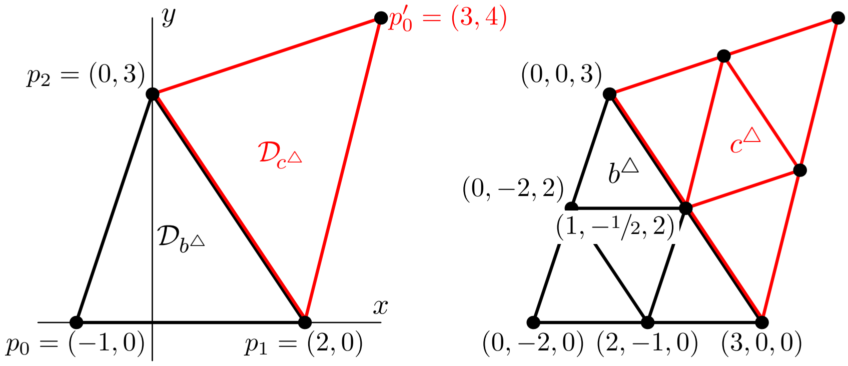

Fig. 3: Domains $\mathcal{D}_{b^\triangle}$ and $\mathcal{D}_{c^\triangle}$ (fig. left) and the coordinates of the control vertices of $b^\triangle$ (fig. right). *Remark:* Coordinates of p'_0, p_0, p_1, p_2 (fig. left) are affine.

8. Assume a tensor-product Bézier patch $\mathcal{P}^\square$ of bidegree $(2,3)$ resp. a cubic triangular Bézier patch $\mathcal{B}^\triangle$ with control vertices

$$\{p_{ij} \mid i = 0,\dots,2; j = 0,\dots,3\} \text{ resp. } \{b_{ijk} \mid i+j+k = 3\}$$

   depicted in fig. 4.

   Determine if the patches are joined in a C^1-continuous manner. Should this not be the case, adjust the coordinates of the point b_{111} so that it is.

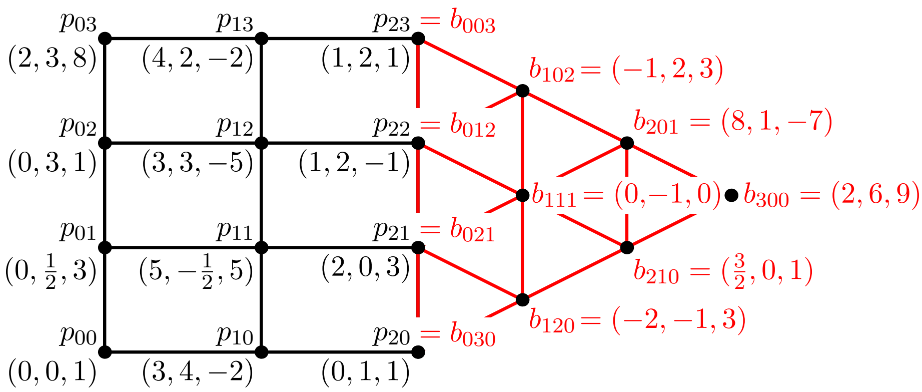

Fig. 4: Control vertices of the patches $\mathcal{P}^\square$ (black) and $\mathcal{B}^\triangle$ (red).

RNDr. Martina Bátorová, PhD.
martina.batorova@fmph.uniba.sk

Bilinear and bicubic Coons patches

9. Assume a bilinear Coons patch

$$\mathcal{S}(s,t) = \mathcal{S}_c(s,t) + \mathcal{S}_d(s,t) - \mathcal{S}_{cd}(s,t) = \begin{pmatrix} st^2 + (s^2-2s+1)t + s \\ st^2 + (s^2-2s+2)t - s - 1 \\ st^3 + (1-s)t^2 + 4(1-s)t - s - 1 \end{pmatrix},$$

with $s, t \in \langle 0, 1\rangle$ and

$$\begin{aligned} \mathcal{S}_c(s,t) &= (1-s)c_0(t) + sc_1(t) \text{ being a ruled surfaces with border curves} \\ & \quad c_0(t) = \begin{pmatrix} t \\ 2t-1 \\ t^2+4t-1 \end{pmatrix} \quad \text{and} \quad c_1(t) = \begin{pmatrix} t^2+1 \\ t^2+t-2 \\ t^3-2 \end{pmatrix}, \\ \mathcal{S}_d(s,t) &= (1-t)d_0(s) + td_1(s) \text{ being a ruled surface with border curves } d_0(s), d_1(s), \\ \mathcal{S}_{cd}(s,t) &= \begin{pmatrix} 1-s & s \end{pmatrix} \cdot \begin{pmatrix} A & C \\ B & D \end{pmatrix} \cdot \begin{pmatrix} 1-t \\ t \end{pmatrix} \text{ being a bilinear patch with corners } A, B, C, D. \end{aligned}$$

Determine the parametrizations of the curves $d_0(s),\ d_1(s)$ and patches $\mathcal{S}_c(s,t),\ \mathcal{S}_d(s,t)$.

Verify the C^0-compatibility of the border curves of$\mathcal{S}(s,t)$.

Prove that $A = c_0(0) = d_0(0),\ C = c_1(0) = d_0(1),\ B = c_0(1) = d_1(0),\ D = c_1(1) = d_1(1)$.

10. Construct a bicubic Coons patch $\mathcal{S}(s,t)$ defined over the domain $\langle 0,1\rangle \times \langle 0,1\rangle$ and interpolating the border curves

$$c_0(t) = \begin{pmatrix} t \\ 0 \\ t-t^2 \end{pmatrix},\ c_1(t) = \begin{pmatrix} t \\ 1 \\ t^3 \end{pmatrix},\ d_0(s) = \begin{pmatrix} 0 \\ s^2 \\ s-s^2 \end{pmatrix},\ d_1(s) = \begin{pmatrix} 1 \\ s \\ s \end{pmatrix}.$$

Verify the C^0-compatibility of the input curves.

Calculate the vector functions $\bar{e}_i(t),\ \bar{f}_i(s), i \in \{0,1\}$ s.t. the output patch is C^2-compatible; work with twists

$$t_{00} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},\ t_{10} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix},\ t_{01} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix},\ t_{11} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

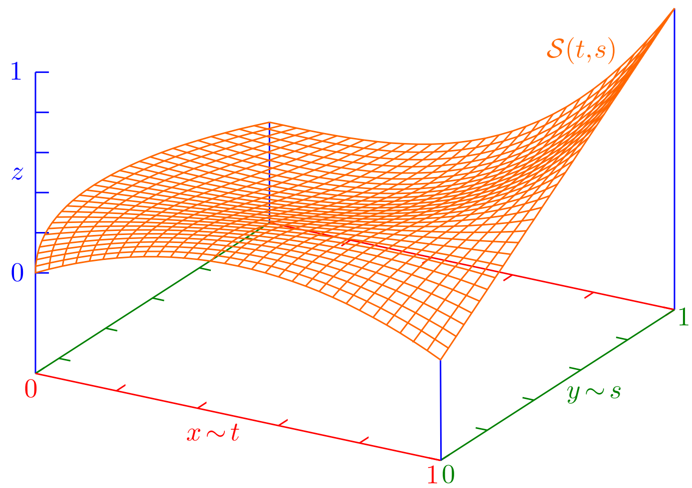

Fig. 5: Output patch $\mathcal{S}(s,t)$.

Remark: The patch $\mathcal{S}(s,t)$ is defined as

$$\mathcal{S}(s,t) = \mathcal{S}_c(s,t) + \mathcal{S}_d(s,t) - \mathcal{S}_{cd}(s,t)$$

with subpatches

$$\begin{aligned}
\mathcal{S}_c(s,t) &= H_0^3(s)c_0(t) + H_1^3(s)\bar{e}_0(t) + H_2^3(s)\bar{e}_1(t) + H_3^3(s)c_1(t),\\
\mathcal{S}_d(s,t) &= H_0^3(t)d_0(s) + H_1^3(t)\bar{f}_0(s) + H_2^3(t)\bar{f}_1(s) + H_3^3(t)d_1(s),\\
\mathcal{S}_{cd}(s,t) &= \begin{pmatrix} H_0^3(s)\\ H_1^3(s)\\ H_2^3(s)\\ H_3^3(s)\end{pmatrix}^{\top} \cdot \begin{pmatrix} c_0(0) & \bar{f}_0(0) & \bar{f}_1(0) & c_0(1)\\ \bar{e}_0(0) & t_{00} & t_{01} & \bar{e}_0(1)\\ \bar{e}_1(0) & t_{10} & t_{11} & \bar{e}_1(1)\\ c_1(0) & \bar{f}_0(1) & \bar{f}_1(1) & c_1(1)\end{pmatrix} \cdot \begin{pmatrix} H_0^3(t)\\ H_1^3(t)\\ H_2^3(t)\\ H_3^3(t)\end{pmatrix},
\end{aligned}$$

and $H_i^3(X), i = 0,\dots,3$ being the cubic Hermite polynomials and $B_i^3(X), i = 0,\dots,3$ being the cubic Bernstein polynomials

$$\begin{aligned}
H_0^3(X) &= & B_0^3(X) + B_1^3(X) &= & 2X^3 - 3X^2 + 1,\\
H_1^3(X) &= & {}^1\!/_3 \cdot B_1^3(X) &= & X^3 - 2X^2 + X,\\
H_2^3(X) &= & -{}^1\!/_3 \cdot B_2^3(X) &= & X^3 - X^2,\\
H_3^3(X) &= & B_2^3(s) + B_3^3(X) &= & -2X^3 + 3X^2.
\end{aligned}$$

The functions $\bar{e}_i(t),\ \bar{f}_i(s),\ i \in \{0,1\}$ are calculated using the following formulae:

$$\begin{aligned}
\bar{e}_i(t) &= H_0^3(t) \cdot \frac{\mathrm{d}\, d_0}{\mathrm{d}\, t}(i) + H_1^3(t) \cdot t_{i0} + H_2^3(t) \cdot t_{i1} + H_3^3(t) \cdot \frac{\mathrm{d}\, d_1}{\mathrm{d}\, t}(i),\\
\bar{f}_i(s) &= H_0^3(s) \cdot \frac{\mathrm{d}\, c_0}{\mathrm{d}\, s}(i) + H_1^3(s) \cdot t_{0i} + H_2^3(s) \cdot t_{1i} + H_3^3(s) \cdot \frac{\mathrm{d}\, c_1}{\mathrm{d}\, s}(i).
\end{aligned}$$

Triangular Coons patches

11. Construct a triangular Coons patch $\mathcal{S}$ over the domain $\mathcal{D} := \triangle p_0p_1p_2$ so that the curves

$$c_0(s) = \begin{pmatrix} 2-s\\ 0\\ 1-s\end{pmatrix},\ c_1(s) = \begin{pmatrix} 2-3s\\ 0\\ 1-2s\end{pmatrix},\ c_2(s) = \begin{pmatrix} 2s-1\\ 0\\ s-1\end{pmatrix},\ s \in \langle 0,1\rangle$$

are its borders and $p_0 = (0,0),\ p_1 = (2,0),\ p_2 = (0,2)$ are the (affine) coordinates of the vertices of $\mathcal{D}$.

Verify the C^0-compatibility of the input data.

Calculate the coordinates of the images of $V, W \in \mathcal{D}$, if their (affine) coordinates are

$$V = ({}^2\!/_3, {}^2\!/_3)\,,\ W = ({}^2\!/_3, {}^1\!/_3)\,.$$

Rational tensor-product Bézier patches

12. Represent the torus using 16 biquadratic rational tensor-product Bézier patches.

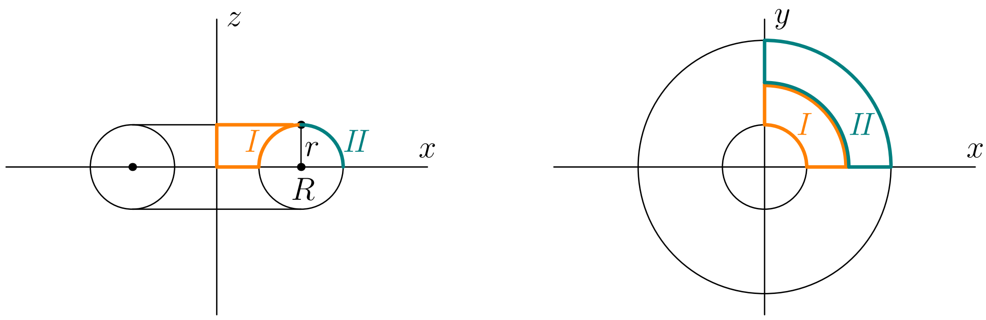

Fig. 6: Projections of the torus into planes $y = 0$ (left) and $z = 0$ (right).

Hint: There are only two different types of patches, I and II (see fig. 6), thus it suffices to calculate the coordinates and weights of their control vertices. Let I and II lie in the first octant, i.e. let $x \geq 0,\ y \geq 0,\ z \geq 0$.

The control vertices on each of the borders and their weights can be calculated using $1/4$-circles. The coordinates of the middle point in each of the patches I, II can be determined using geometrical ideas and properties e.g. the vertex is a point of the plane $z - r = 0$ (why?), and thus $V_{11} = (x_{11}, y_{11}, r)$. Its weight can be determined using weights $w_0 = w_2 = \frac{1}{\sqrt{2}}$ and the fact that the ratio of the weights of the vertices within given curve is preserved (why?).

Naturally, the coordinates of V_{11} can be calculated using the de Casteljau algorithm for $(u, v) = (1/2, 1/2)$, since the coordinates of the image of $(1/2, 1/2)$ are easily available using e.g. the parametrization of the torus $\tau(\varphi, \theta)$; this parametrization can be obtained e.g. using the sweeping procedure and a subsequent substitution of suitable values of $\varphi, \theta \in \langle 0, 2\pi)$ (which ones?).

13. Represent the torus using 4 biquadratic rational Bézier patches; here, also zero weights are permitted. All the patches are identical, thus it suffices to determine the coordinates only for one of them, e.g. when $y \geq 0$ and $z \geq 0$ (see fig. 7).

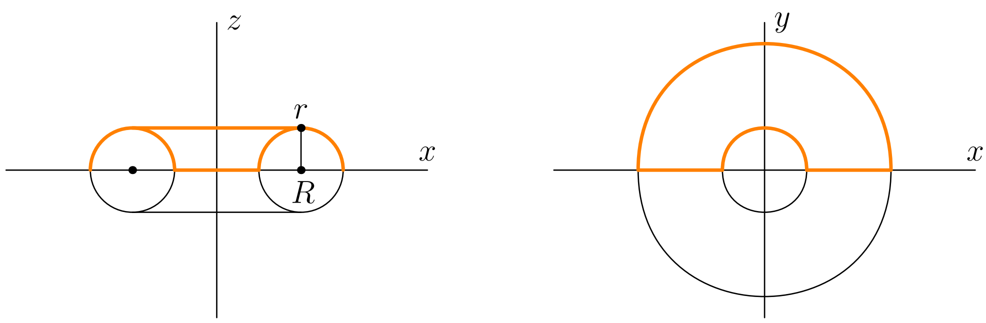

Fig. 7: Projections of the torus into planes $y = 0$ (left) and $z = 0$ (right).

14. Assume an ellipse $\mathcal{E}$ in the plane $z = 0$ with centre in the origin $(0, 0, 0)$, with a being the length of the major and b the length of the minor axis, and e being the excentricity i.e. the distance between each of the focal points and the centre (see fig. 8 left).

    Let us construct the cyclide $\mathcal{C}$ (see fig. 9) using the ellipse $\mathcal{E}$ and a rope of length $|a + k|$ (with $k > 0$ being a suitable parameter) in the following fashion. Let us pin one end of the tightened rope into the focal point $(-e, 0, 0)$ of $\mathcal{E}$ and let it move along $\mathcal{E}$. The free end of the rope matches the surfaces of $\mathcal{C}$.

    Represent the cyclide using four identical rational biquadratic Bézier patches. As in the torus case, it suffices to determine the vertices and weigths only for the part of $\mathcal{C}$ with $y \geq 0$ and $z \geq 0$.

    What are the values of a, e, k so that the torus resp. a sphere is obtained?

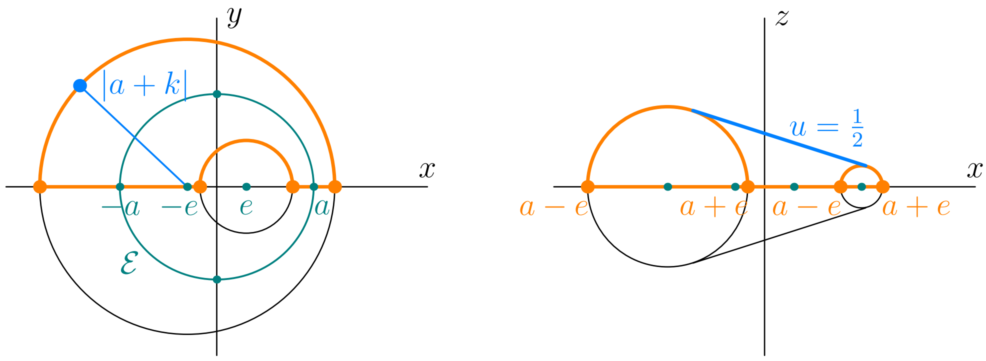

Fig. 8: Projections of the cyclide to plane. On the left, the focal points and vertices of the ellipse $\mathcal{E}$ are depicted. On the right, the weights of the corner points are given.

Hint: The construction above is useful to calculate the coordinates of the corner points, the rest of the vertices can be determined using previous exercises and procedures. In order to determine the weights of the corner points, use those given in fig. 8 (right).

The rest of the vertices can be determined using the implicit equation of the cyclide:

$$\mathcal{C}(x, y, z)\colon\ \left(x^2 + y^2 + z^2 + a^2 - e^2 - k^2\right)^2 - 4\left((a^2 - e^2)y^2 + (ax - ek)^2\right) = 0,$$

resp. its parametrization:

$$\gamma(\varphi, \theta) = \frac{1}{a - e\cos\varphi\cos\theta} \cdot \begin{bmatrix} k(e - a\cos\varphi\cos\theta) + (a^2 - e^2)\ \cos\varphi \\ \sqrt{a^2 - e^2}\,(a - k\cos\theta)\ \sin\varphi \\ \sqrt{a^2 - e^2}\,(k - e\cos\varphi)\ \sin\theta \end{bmatrix}, \quad \varphi, \theta \in \langle 0, 2\pi\rangle.$$

The coordinates of the remaining control vertex p_{11} and its weight are calculated in the same way as in the torus case in exercise 13. Use the values $u = 1/2$ resp. values $\theta = \frac{\pi}{2},\ \varphi \in \langle 0,\ \pi\rangle$.

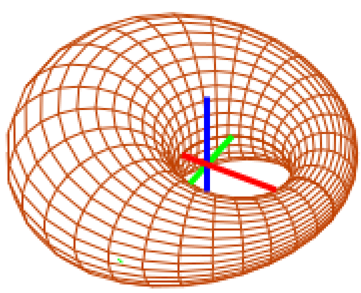

Fig. 9: Cyclide as 4 rational Bézier patches.

RNDr. Martina Bátorová, PhD.
martina.batorova@fmph.uniba.sk

B-splines and NURBS

15. Assume the quadratic B-spline functions $\{N_i^2(u) \mid i = 0, 1, 2\}$ defined over the knot vector $\mathcal{U} := \langle u_0, u_1, u_2, u_3, u_4, u_5\rangle$.

    Construct the B-spline functions $\{N_i^2(u)\}$ over both the uniformed $\mathcal{U} = \langle 0, 1, 2, 3, 4, 5\rangle$ and non-uniformed knot vector $\mathcal{U}$ containing multiple knots. Draw their graphs.

    Determine the $\{N_i^2(u)\}$ functions defined over the knot vector $\mathcal{U} = \langle 0, 0, 0, 1, 1, 1\rangle$ and interpret the result.

16. Assume a B-spline curve $\mathcal{B}(u)$ with control vertices $\langle V_0, \ldots, V_4\rangle$ defined over the knot vector

    $$\mathcal{U} := \langle 0, 0, 0, 1, {}^5/_2, 3, 3, 3\rangle \,.$$

    Determine the number of segments of the spline $\mathcal{B}(u)$ and a degree of each of those segments. Draw its graph.

17. (*NURBS circle of nine points*) Represent the circle $k = [(0, 0); r]$ as a quadratic non-uniformed rational B-spline curve (NURBS) with control vertices $\langle V_0, \ldots, V_8 \mid V_0 = V_8\rangle$. These vertices are distributed uniformly along a square s.t. $V_0 = V_8$.

    Hint: One should assign a set of weights $\langle w_0, \ldots, w_8 \mid w_0 = w_8\rangle$ to $\langle V_i \mid i = 0, \ldots, 8\rangle$ and construct a suitable knot vector $\mathcal{U}$ so that the output NURBS curve interpolates the vertices V_i.

18. Represent a circular cylinder as a NURBS patch of bidegree $(2, 1)$.

    Hint: Use exercise 17 and determine a knot vector $\mathcal{V}$ corresponding to the v parameter s.t. the output NURBS patch interpolates the control vertices of both bases of the cylinder.