Transformácie obrazu

Gonzales, Woods: Digital Image Processing
kapitola: Image transforms

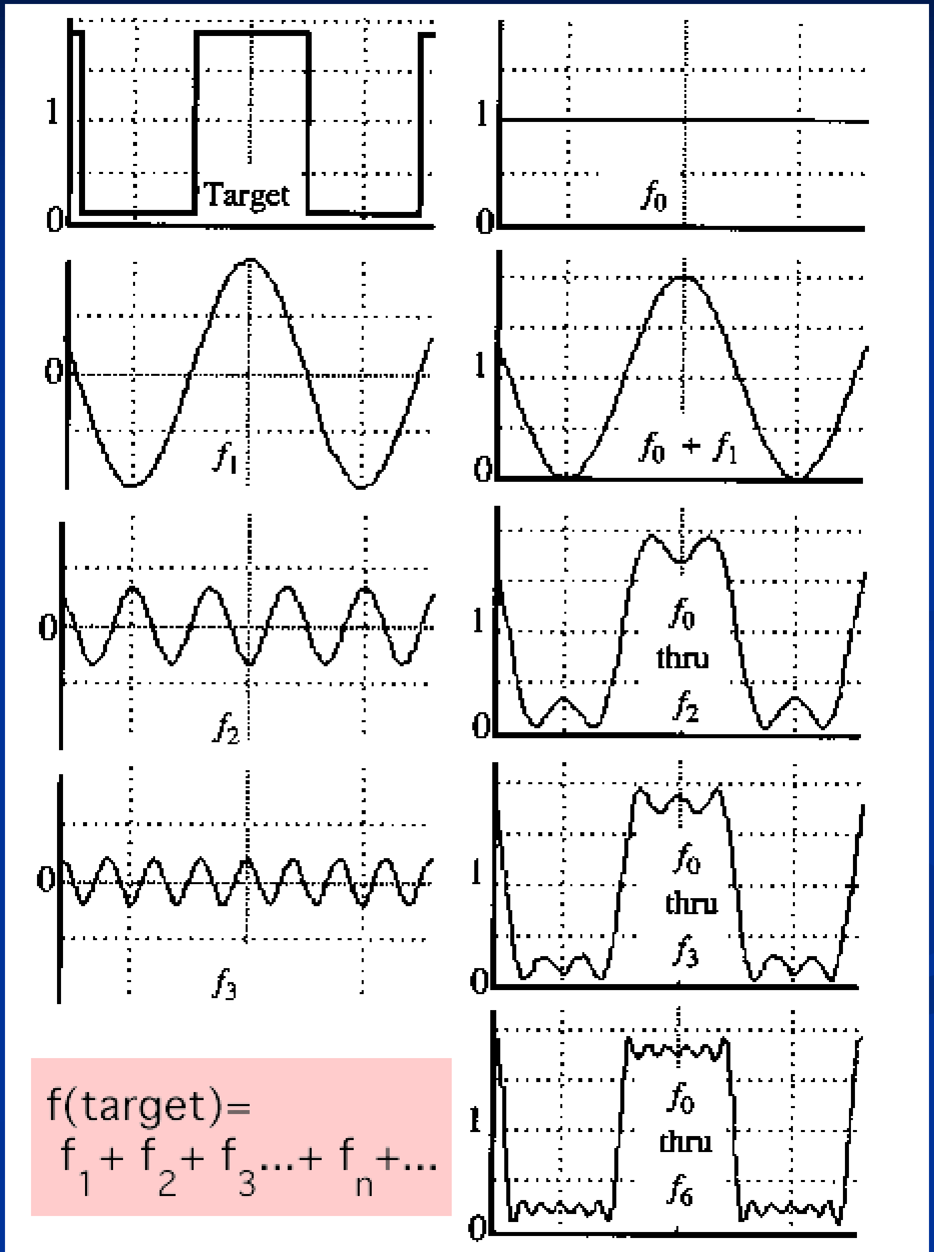
Fourierova transformácia

- Jean Baptiste Joseph Fourier (1768-1830)

- Akákoľvek funkcia f(x) môže byť vyjadrená ako vážený súčet sínusov a kosínusov

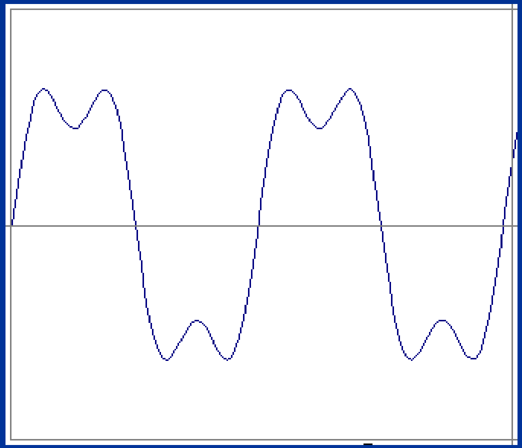
Suma sínusov a kosínusov

- $A \sin(\omega x + \varphi)$



Time and Frequency

- example : $g(t) = \sin(2\pi f\, t) + (1/3)\sin(2\pi(3f)\, t)$



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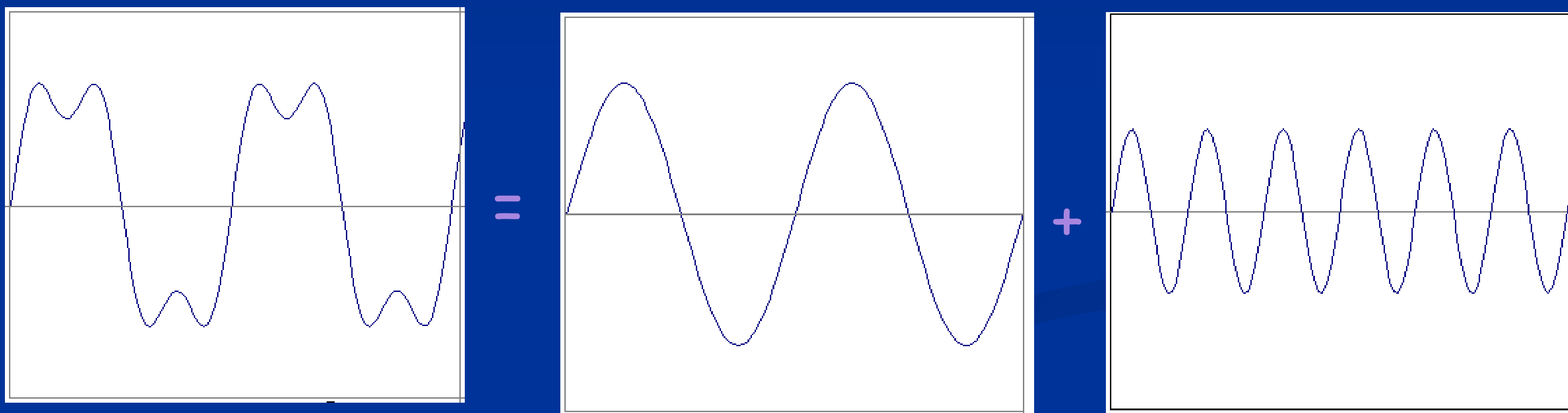

Frequency Spectra

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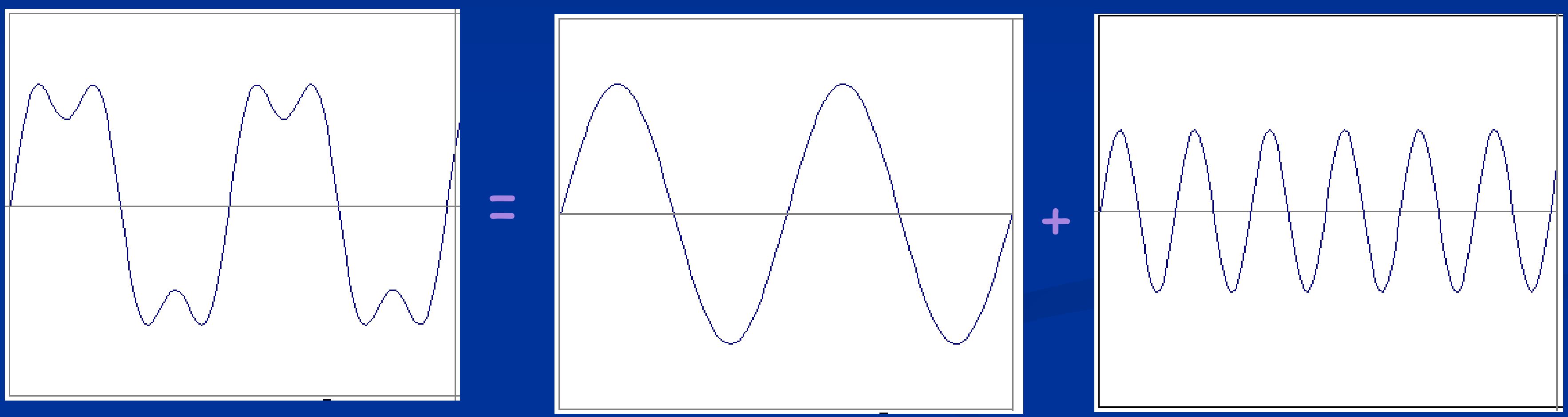
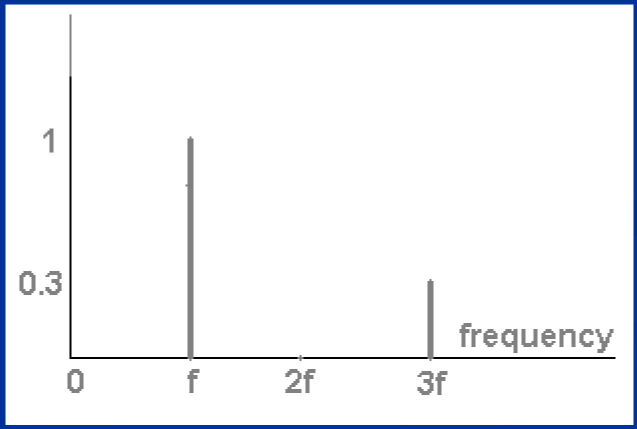
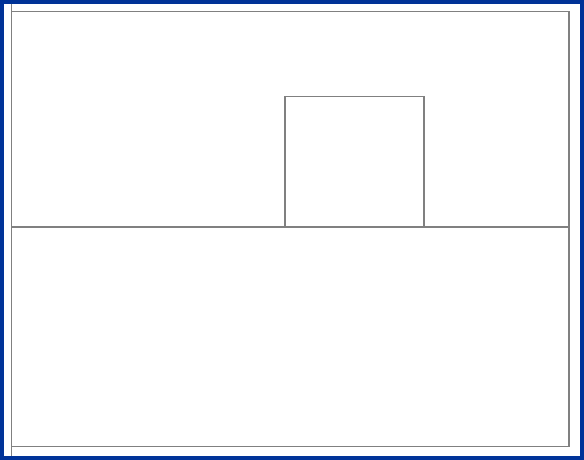
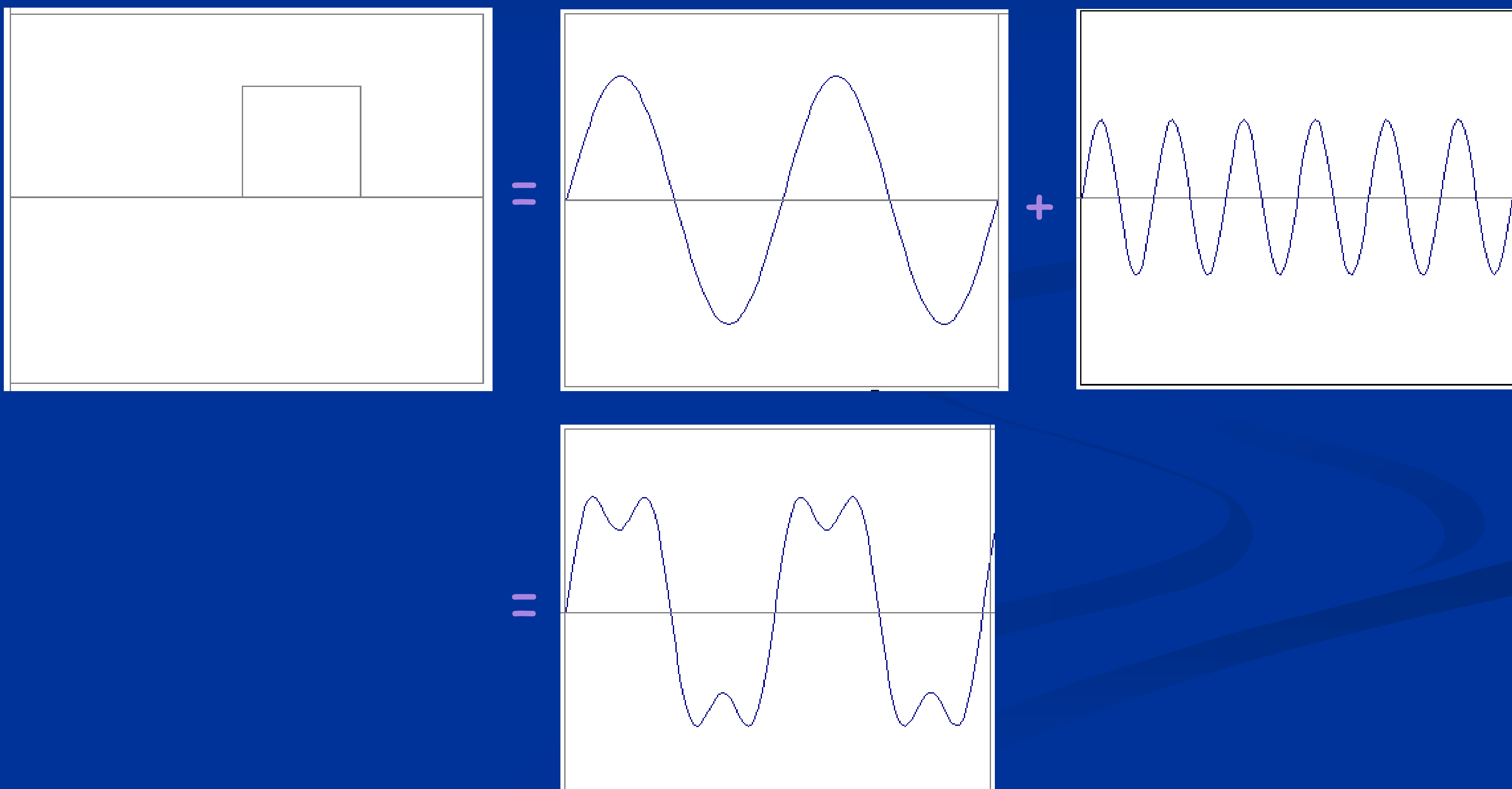




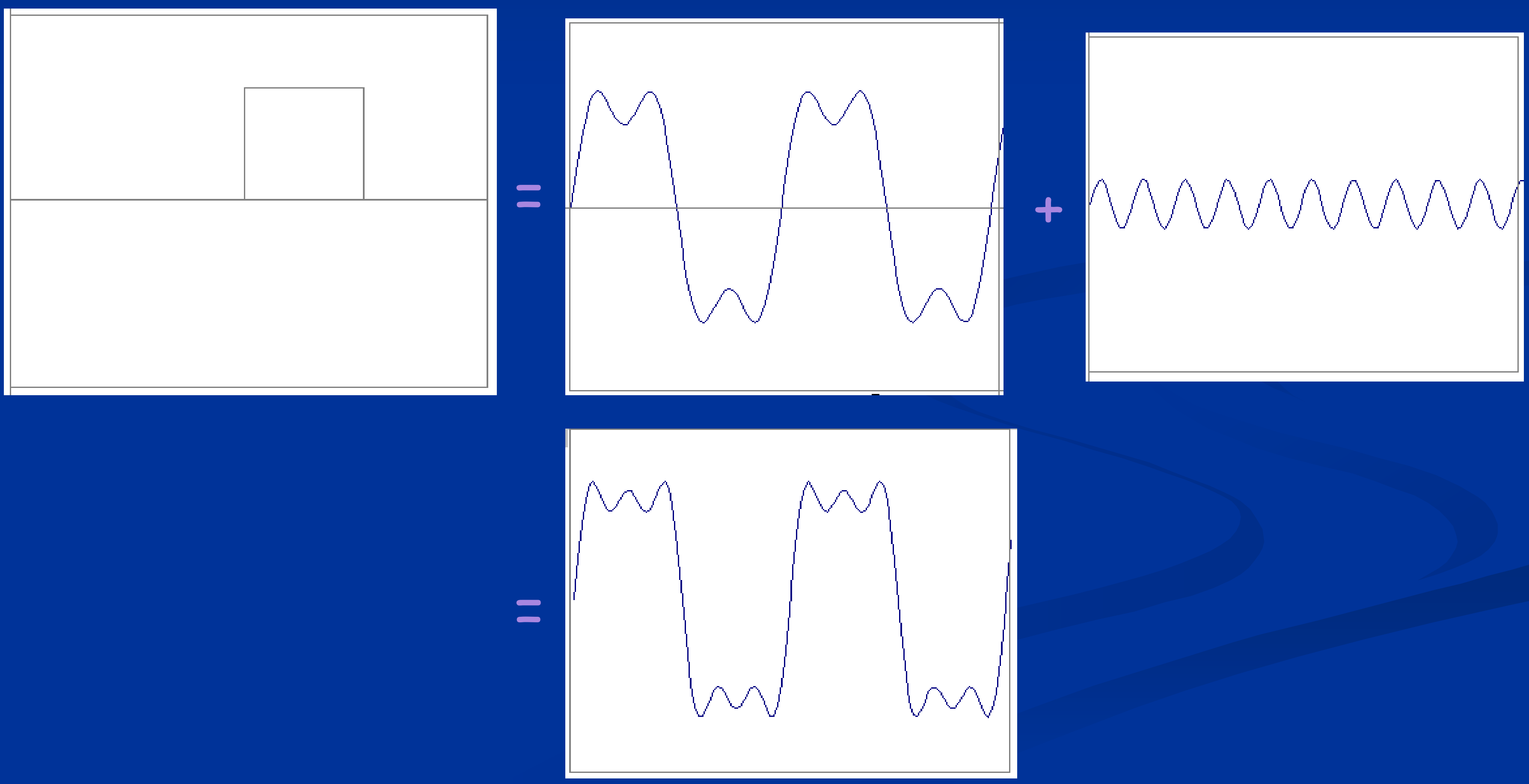
Frequency Spectra



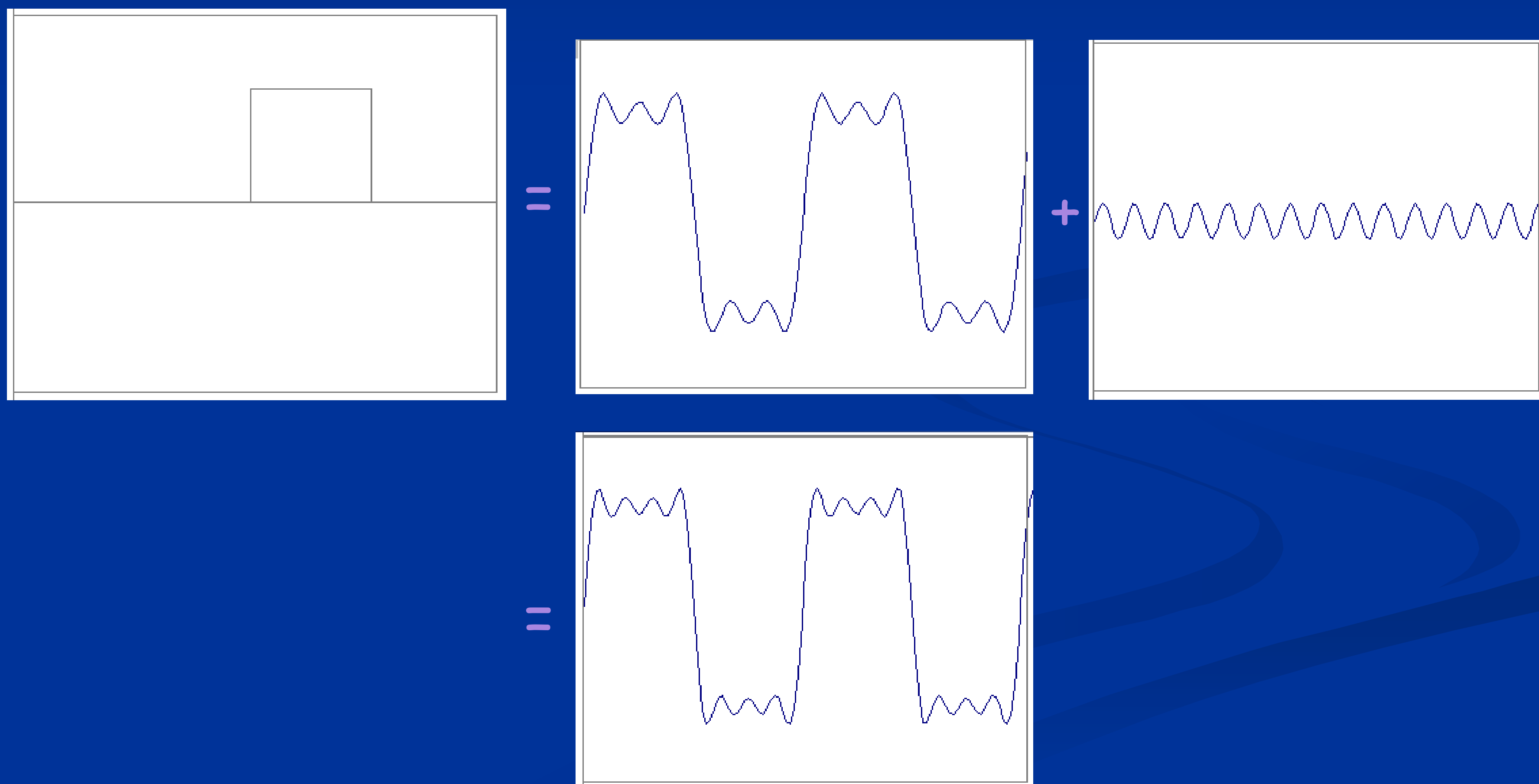
Frequency Spectra



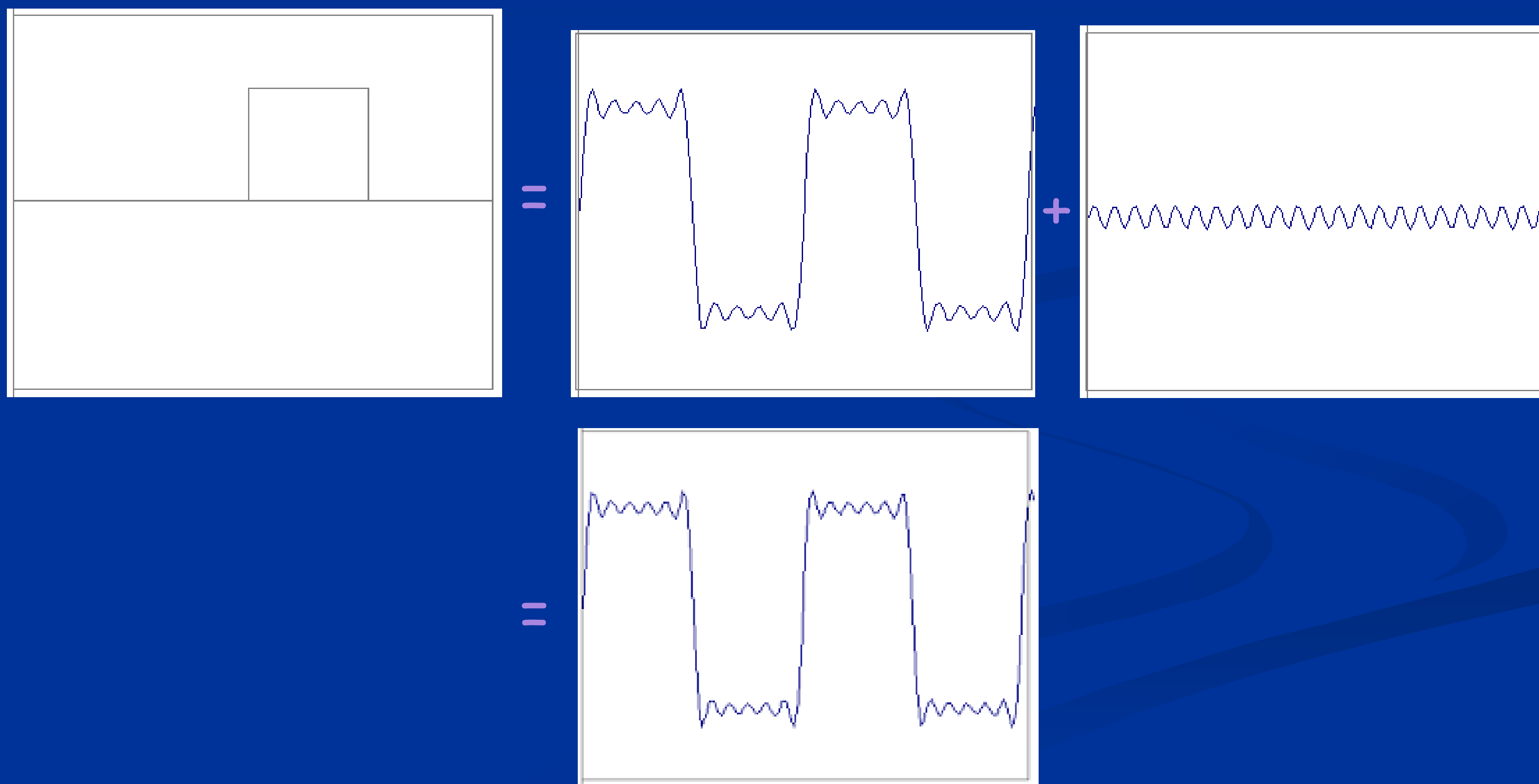
Frequency Spectra



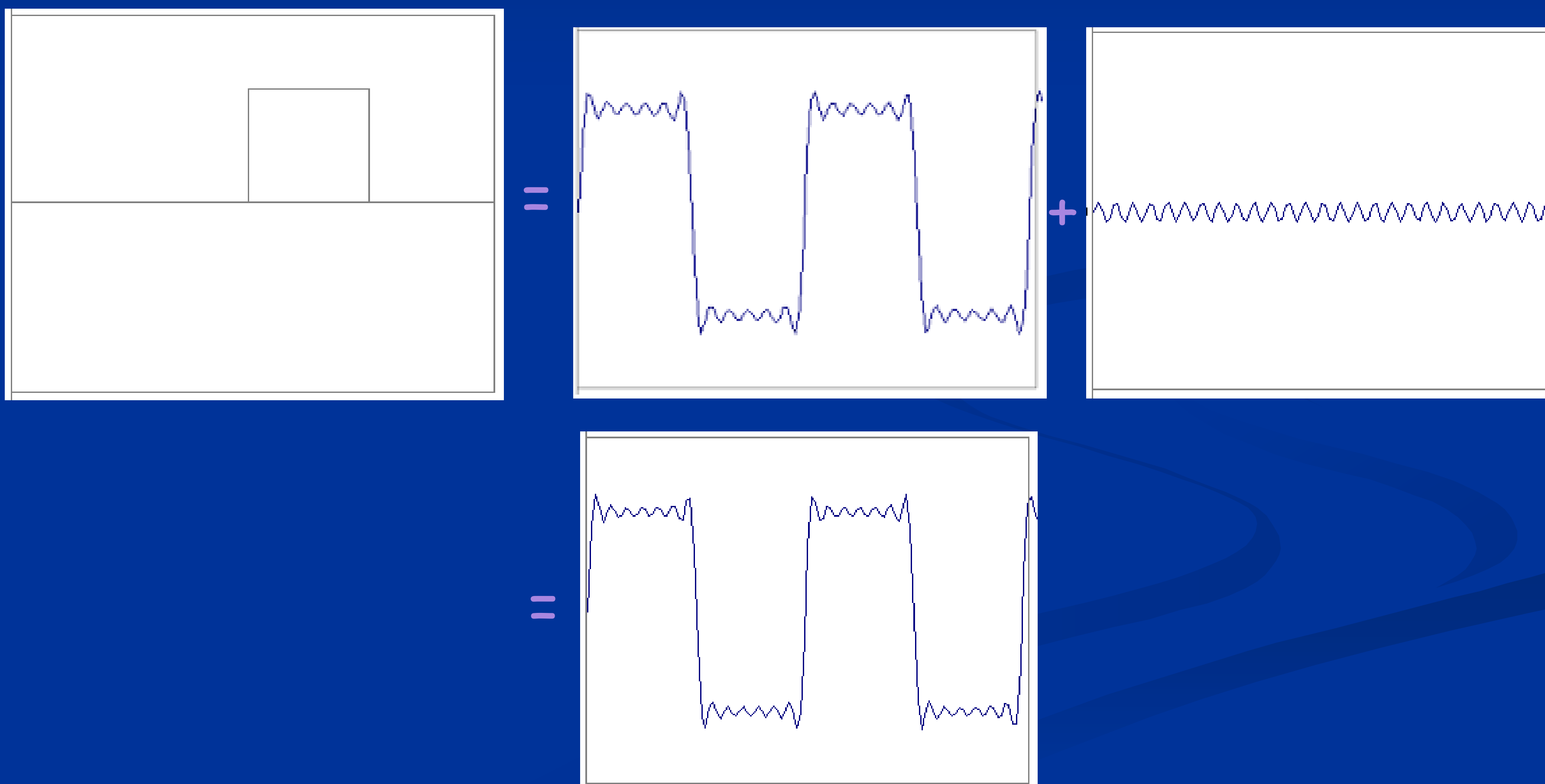
Frequency Spectra



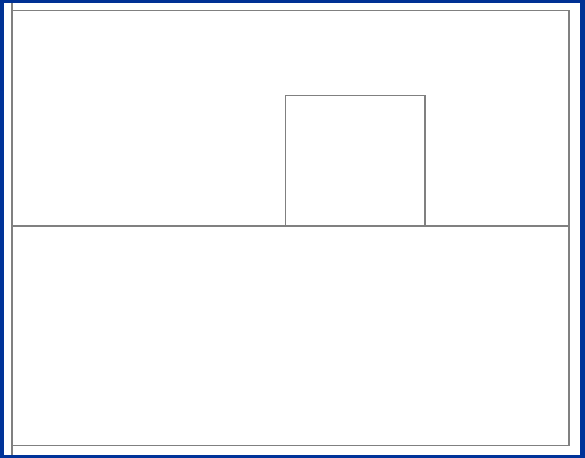
Frequency Spectra



Frequency Spectra

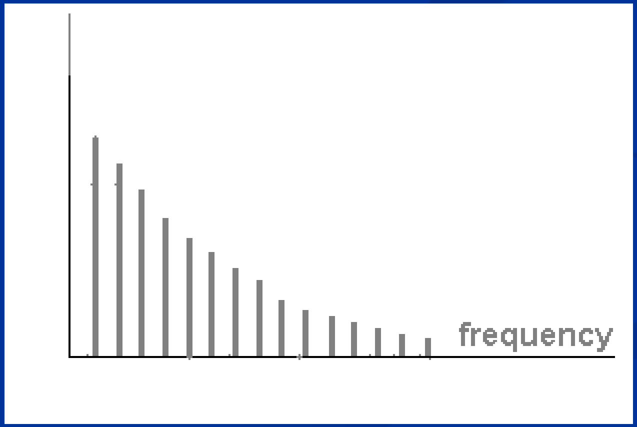


Frequency Spectra

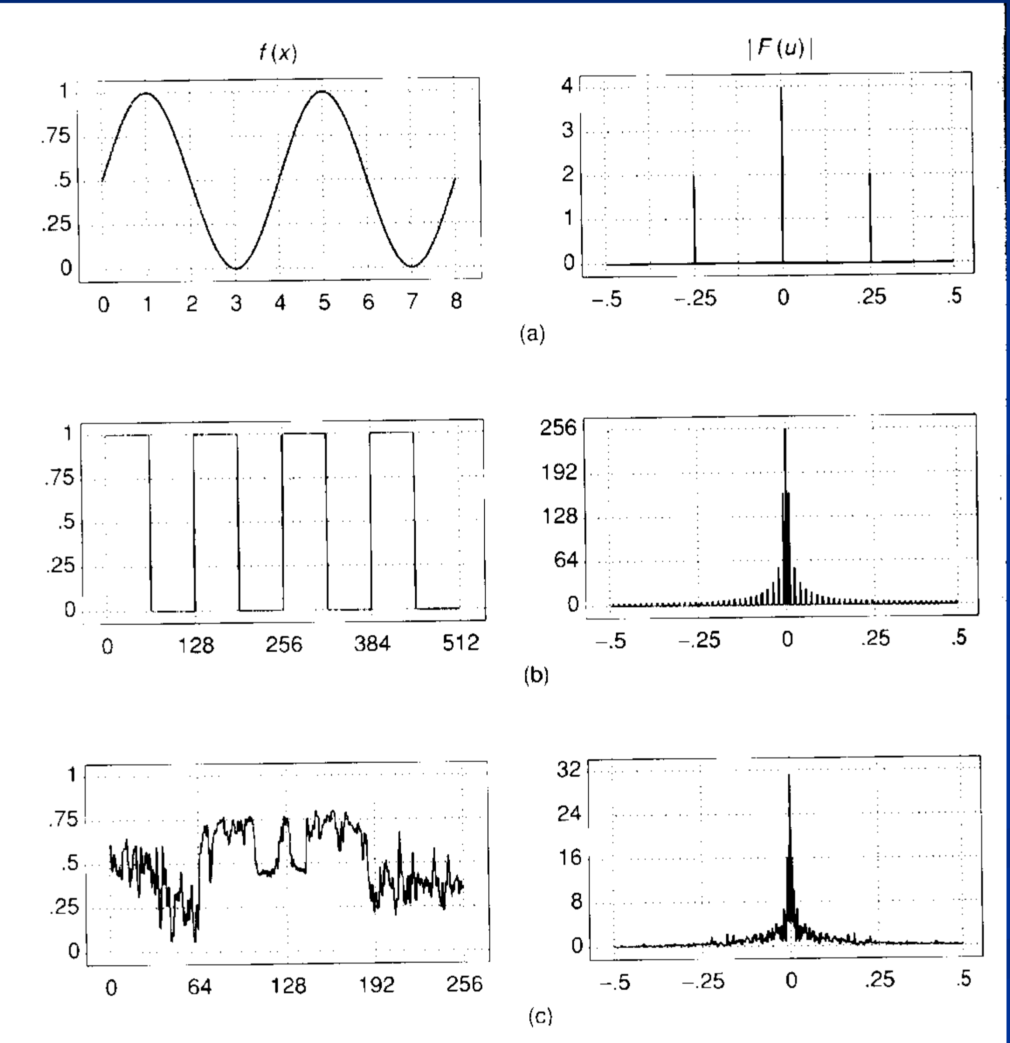


=

$$A\sum_{k=1}^{\infty}\frac{1}{k}\sin(2\pi kt)$$

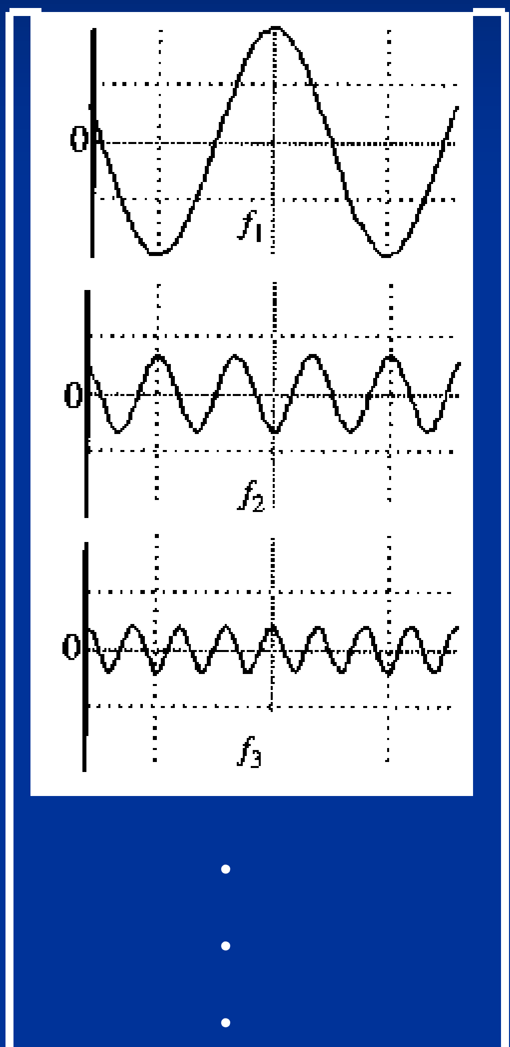


Frequency Spectra



FT: Just a change of basis

$$M * f(x) = F(\omega)$$



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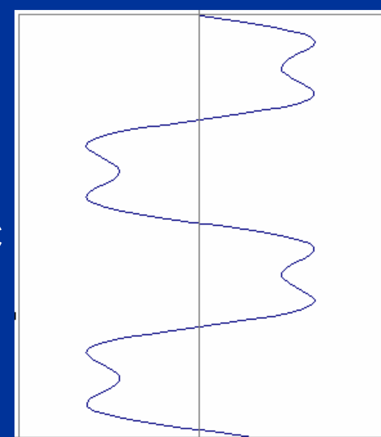

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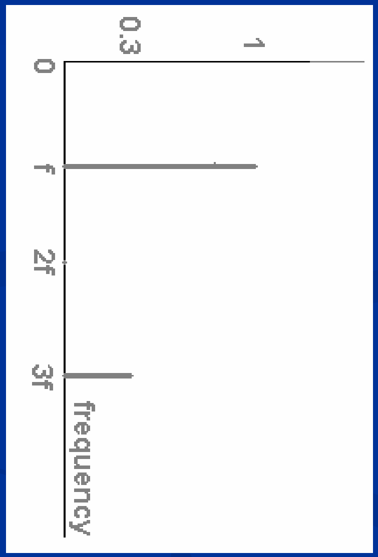
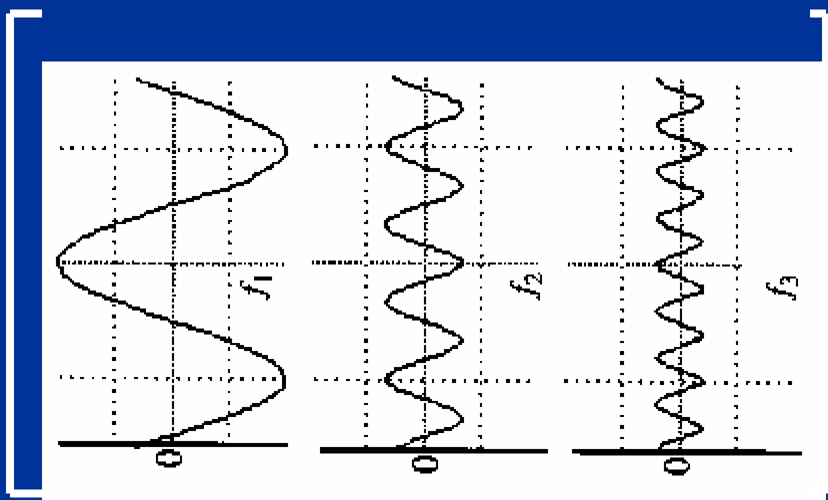


IFT: Just a change of basis

$$M^{-1} * F(\omega) = f(x)$$



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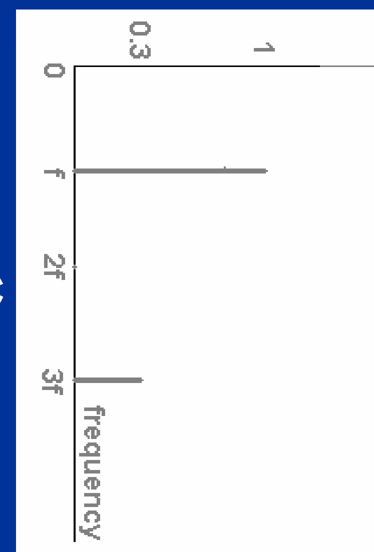

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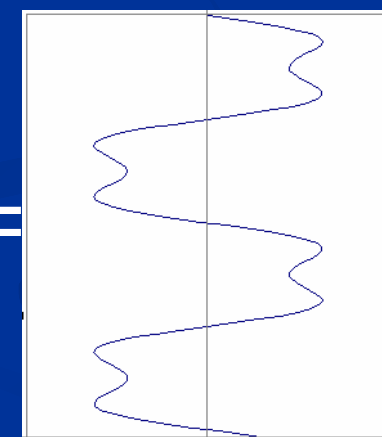
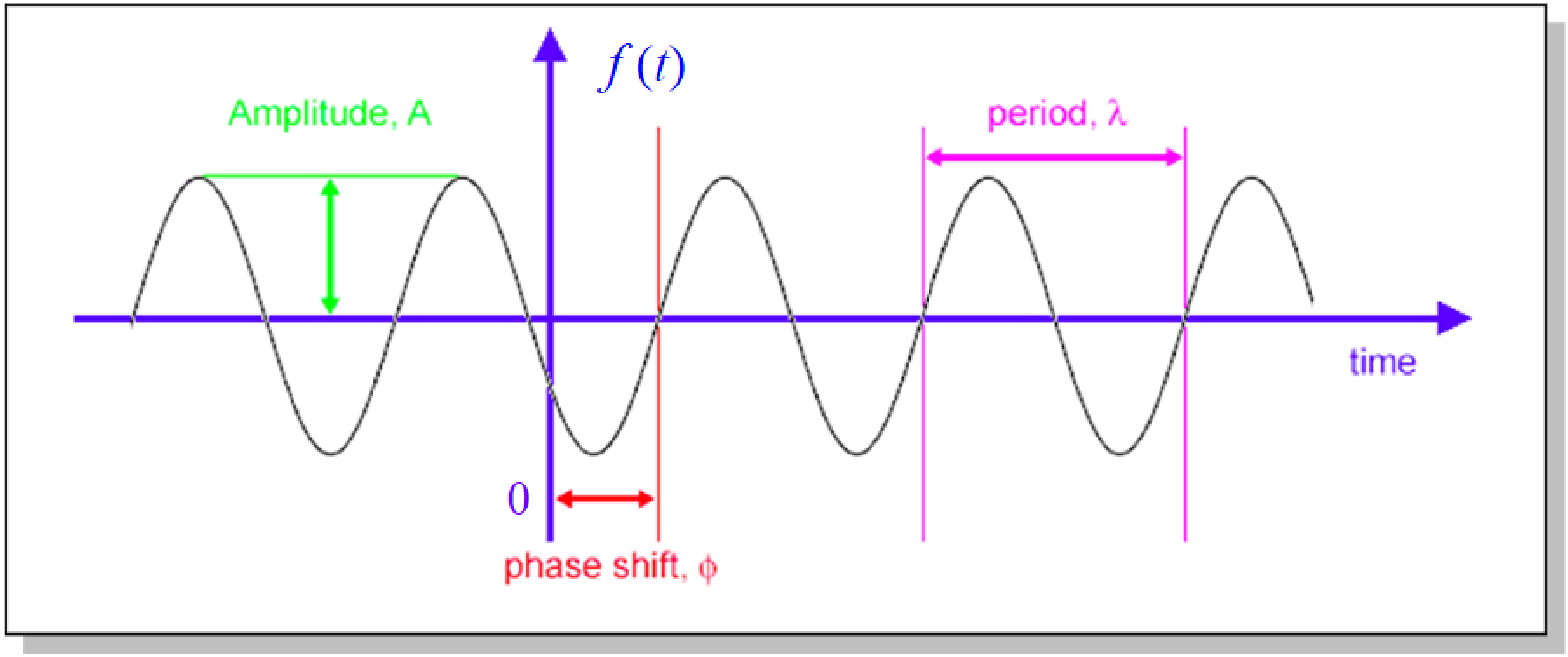




$$f(t) = A\sin\left(\frac{2\pi}{\lambda}t - \phi\right)$$

$1/\lambda$ is the frequency of the sinusoid (Hz).
$2\pi/\lambda$ is the angular frequency (radians/s).

Fourierova transformácia

Ak *f(x)* je spojitá funkcia s reálnou premenou *x*, potom Fourierovou transformáciou *F(x)* je

$$F(u) = \int_{\infty}^{-\infty} f(x) e^{-i2\pi ux} dx,$$

$$F(u) = \int_{\infty}^{-\infty} f(x)(\cos 2\pi ux - i \sin 2\pi ux) dx.$$

Inverznou Fourierovou transformáciou nazývame

$$f(x) = \int_{\infty}^{-\infty} F(u) e^{i2\pi ux} du.$$

pár Fourierovej transformácie

Fourier Transform

f(x) → [Fourier Transform] → ***F(u)***

Pre každé *u* od 0 po inf, ***F(u)*** obsahuje amplitudu A and fázu ϕ odpovedajúceho sinusu

$$A\sin(ux+\phi)$$

$$F(u) = R(u) + iI(u),$$

$$A = \pm\sqrt{R(u)^2 + I(u)^2}$$

$$\phi = \tan^{-1}\frac{I(u)}{R(u)}$$

F(u) → [Inverse Fourier Transform] → ***f(x)***

Definitions

- ***F(u)*** sú komplexne čísla: $F(u) = R(u) + iI(u),$
  $F(u) = r(\sin\theta + i\cos\theta) = re^{i\theta}$
- Magnitúda FT (spektrum): $|F(u)| = \left[R^2(u) + I^2(u)\right]^{1/2}$
- Fázový uhol FT: $\phi(u) = \tan^{-1}\left[\frac{I(u)}{R(u)}\right]$
- Reprezentácia pomocou magnitúdy a fázy: $F(u) = |F(u)|e^{i\phi(u)},$
- Power of f(x): **P(u)**=|F(u)|^2= $R^2(u) + I^2(u)$

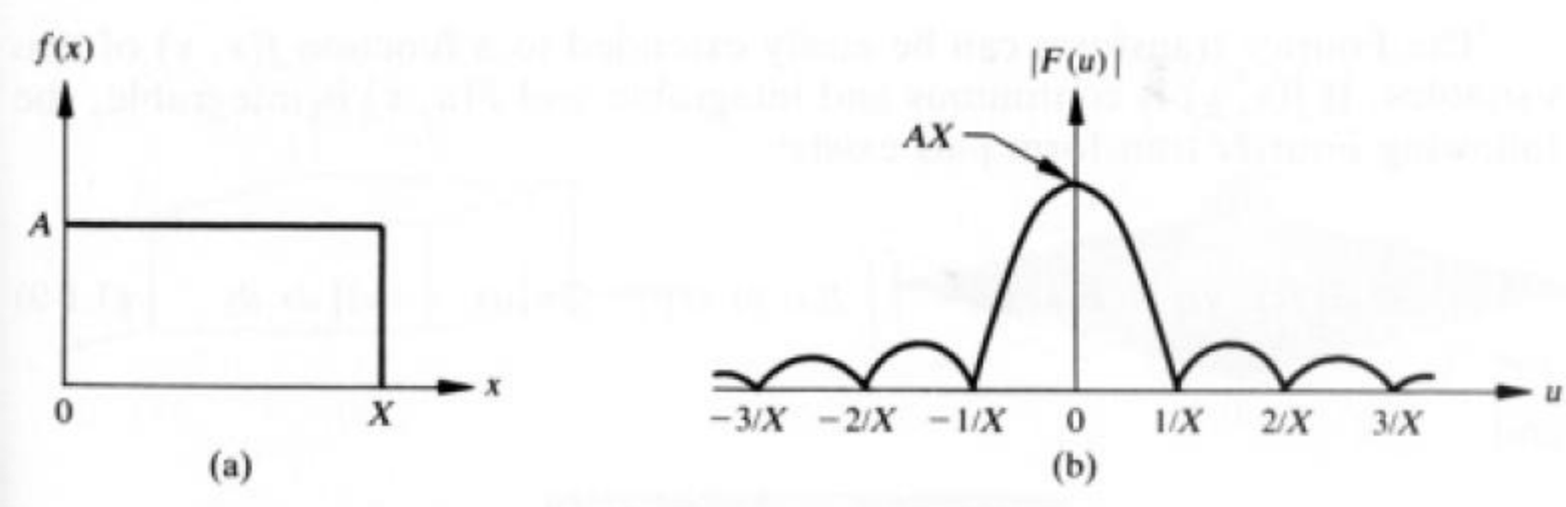


Figure 4: A simple function and its Fourier spectrum.

$$F(u) = \frac{A}{\pi u} \sin(\pi u X) e^{-i\pi u X}$$

$$|F(u)| = AX \left| \frac{\sin(\pi u X)}{(\pi u X)} \right| .$$

FT – je periodická s periódou N, to znamená, že na jej určenie stačí jedna perióda vo frekvenčnej oblasti.

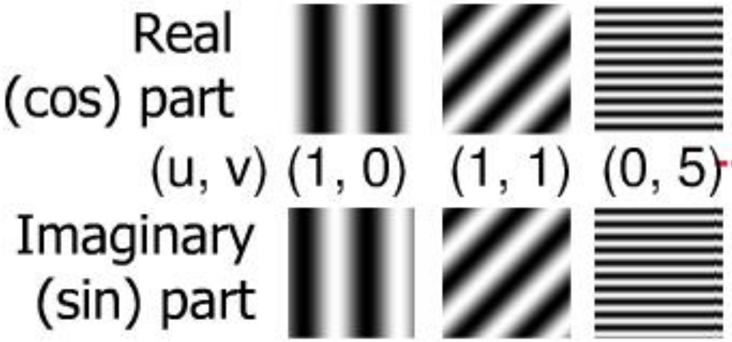
2D Fourierova transformácia

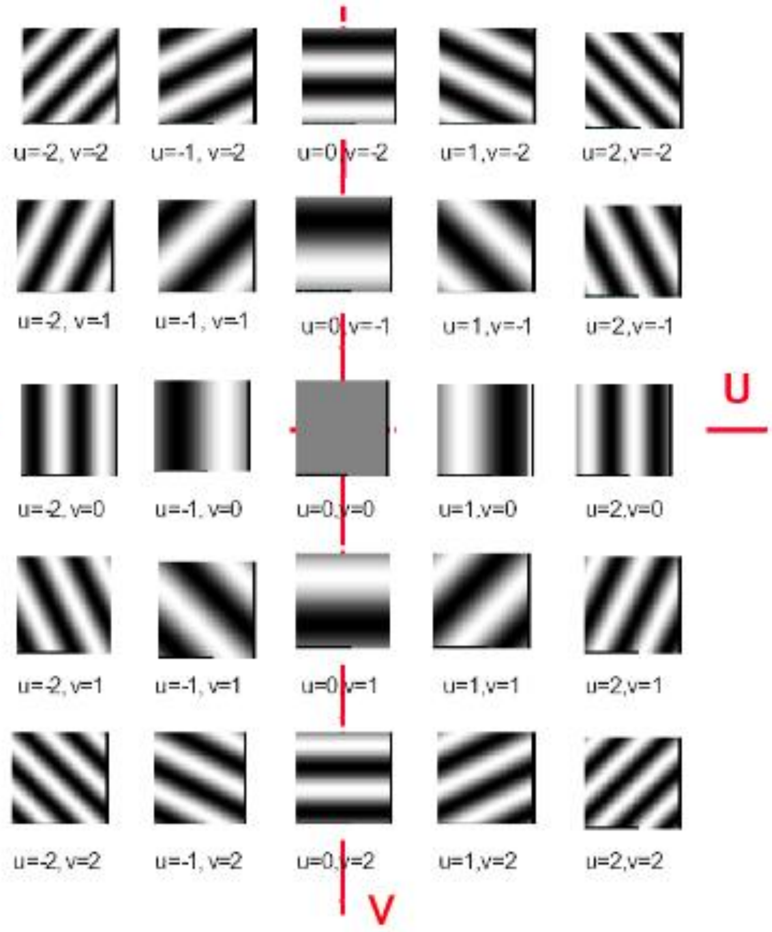
- Fourierova transformáci je ľahko rozšíriteľná do 2D

$$F(u,v) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\exp\left[-i\,2\pi(ux+vy)\right]dx\,dy$$

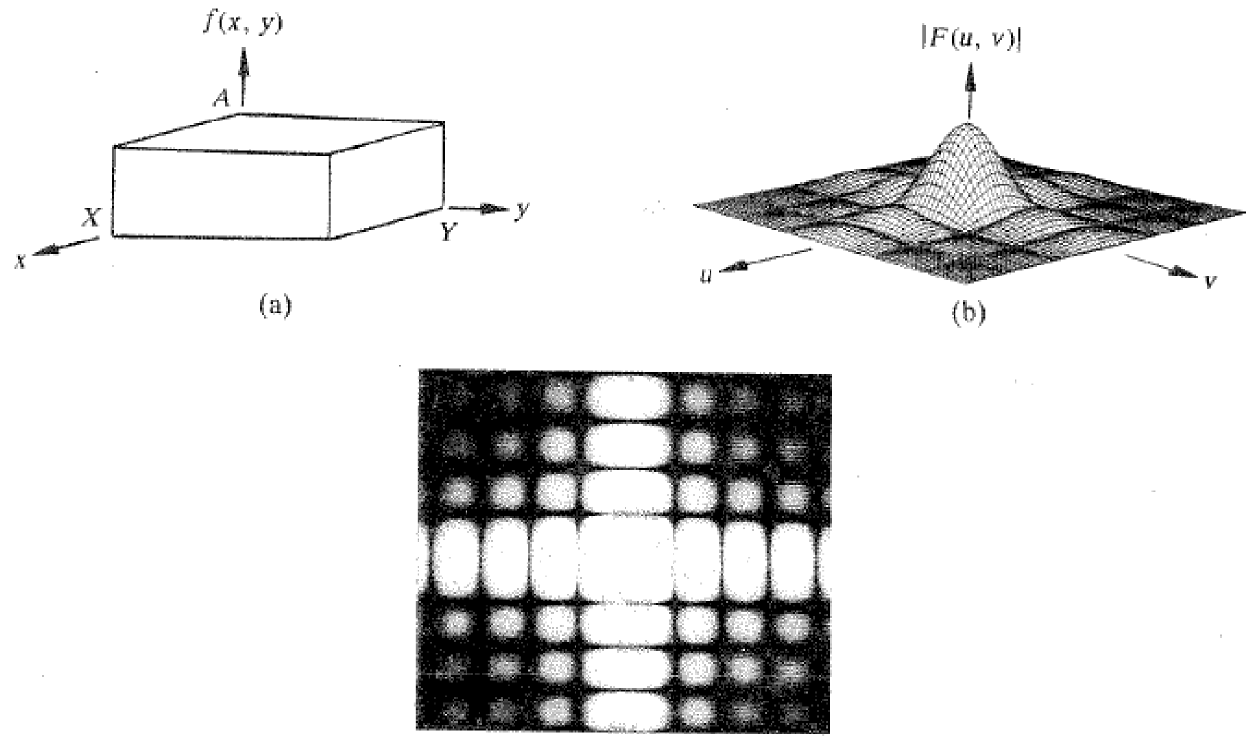
$$f(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(u,v)\exp\left[i\,2\pi(ux+vy)\right]du\,dv\,.$$

Fourier Basis





Príklad 2D funkcie

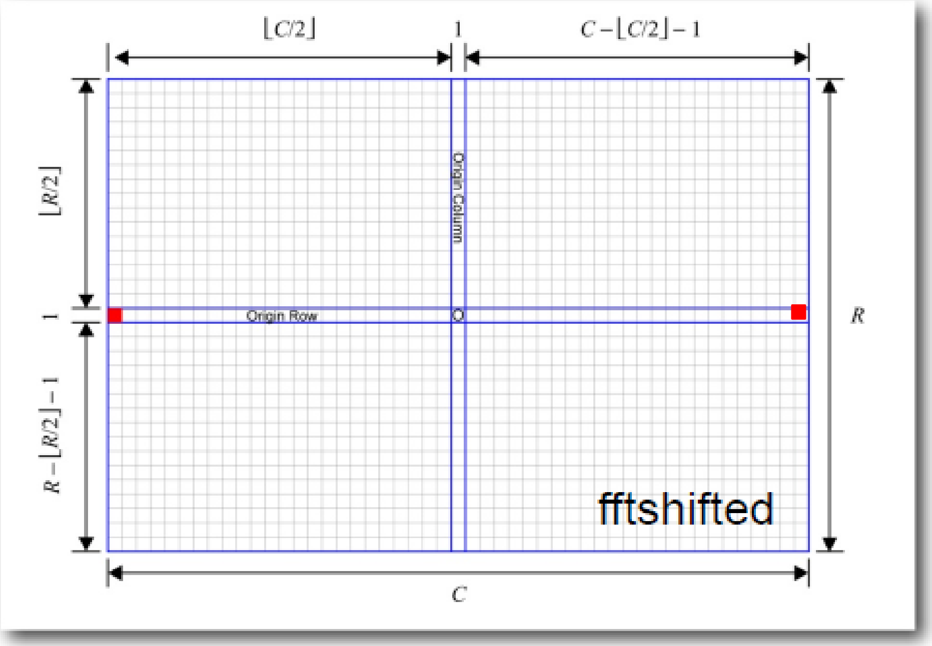


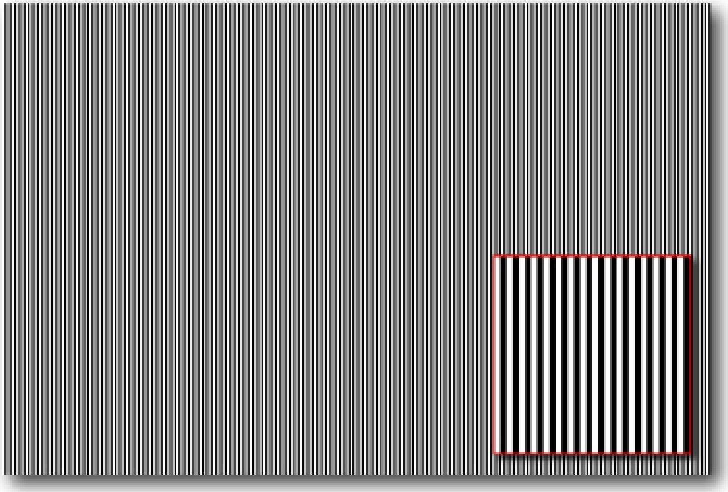
(a) (b) (c)

$$F(u,v)AXY\left[\frac{\sin(\pi uX)e^{-i\pi uX}}{(\pi uX)}\right]\left[\frac{\sin(\pi vY)e^{-i\pi vY}}{(\pi vY)}\right]$$

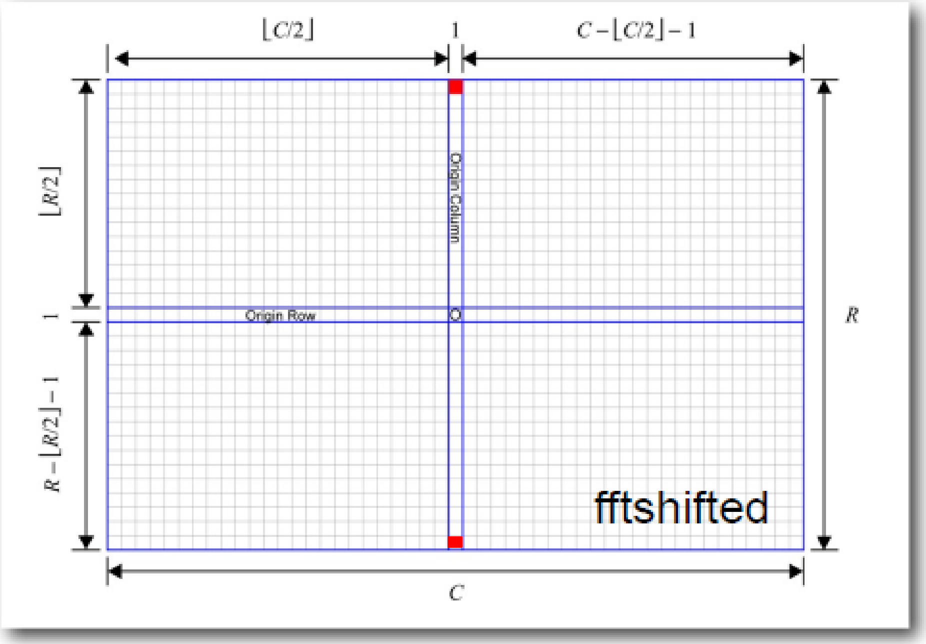
$$\left|F(u,v)\right| = AXY\left|\frac{\sin(\pi uX)}{(\pi uX)}\right|\left|\frac{\sin(\pi vY)}{(\pi vY)}\right|$$

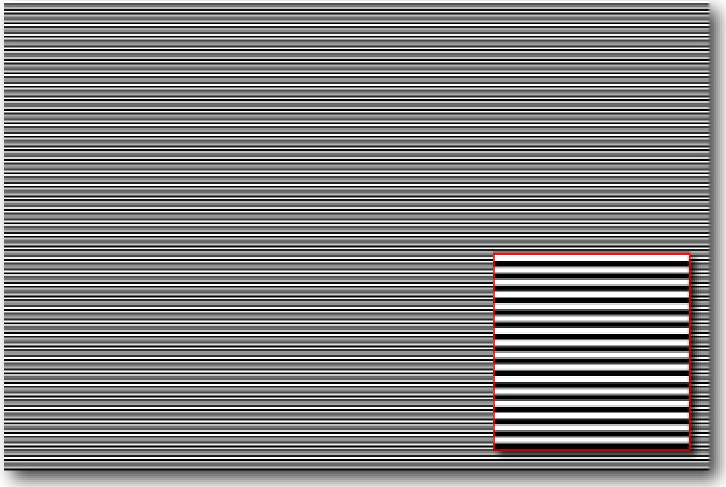
“horizontal” is the wavefront direction.



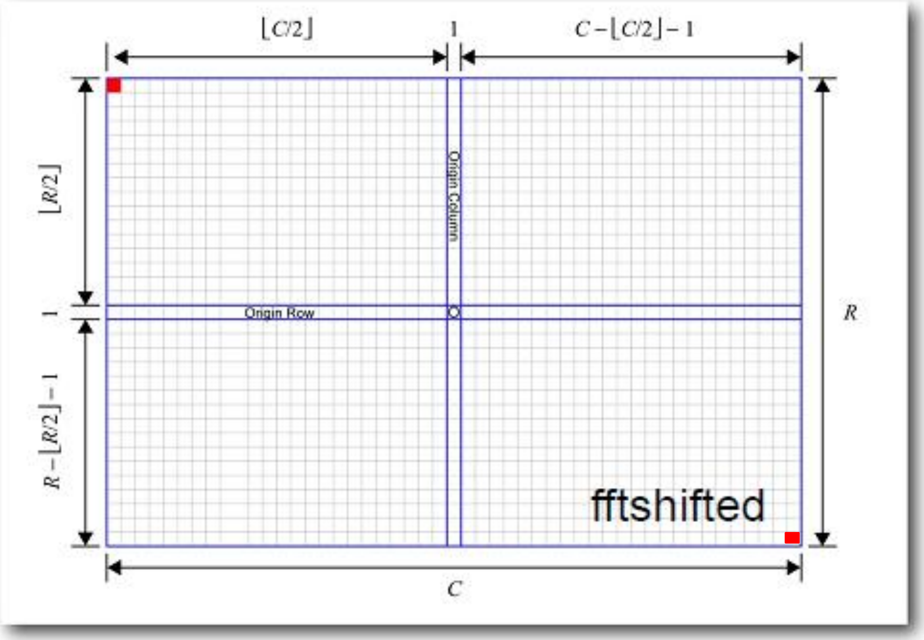


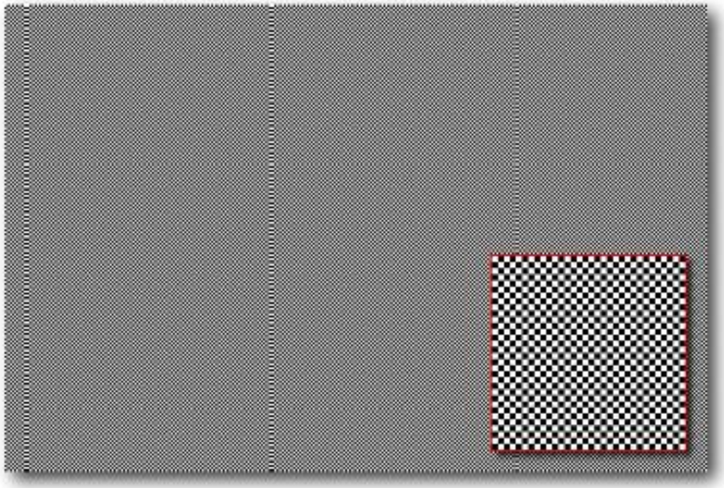
highest-possible-frequency horizontal sinusoid “vertical” is the wavefront direction.



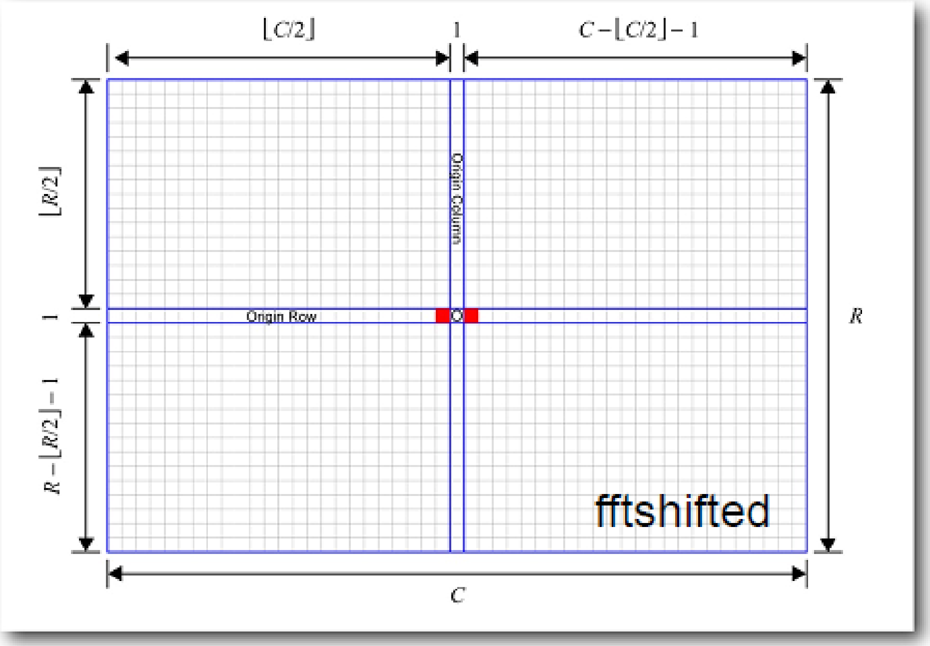


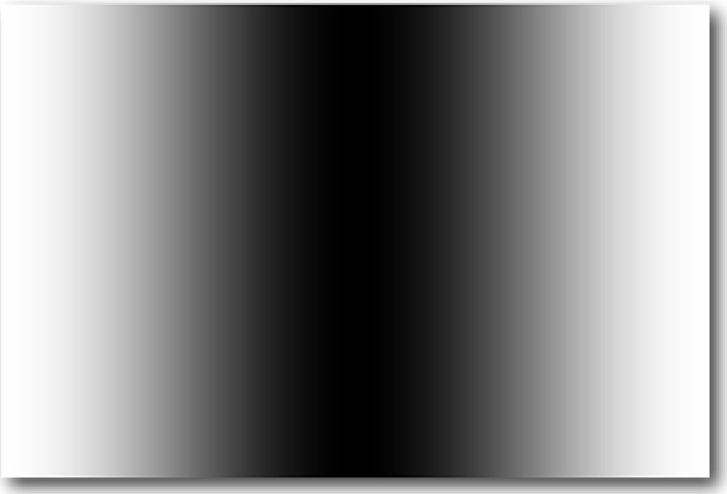
highest-possible-frequency vertical sinusoid a checker-board pattern.



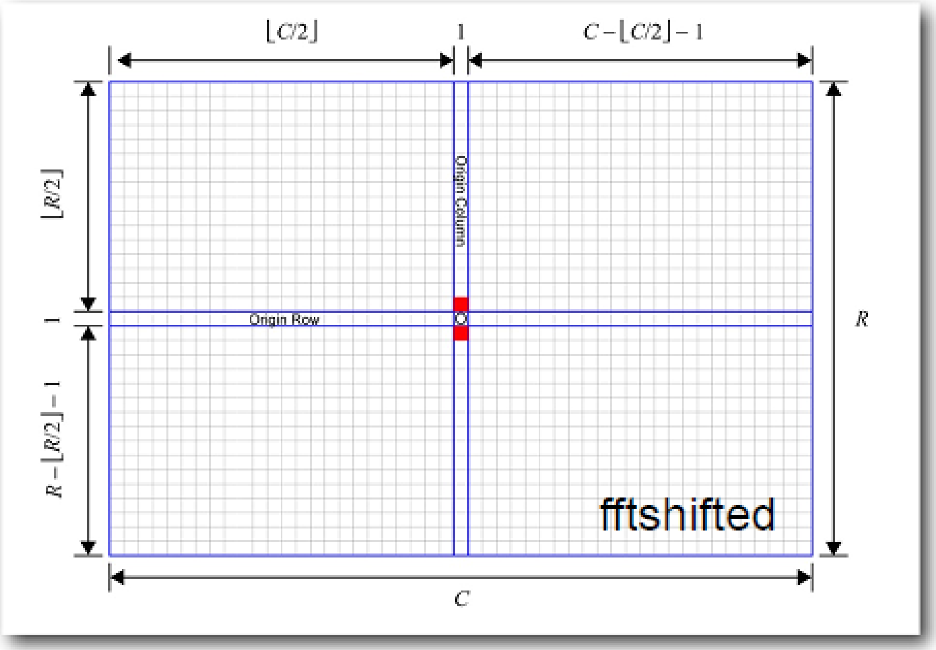


highest-possible-freq horizontal+vertical sinusoid (*R* & *C* even) "horizontal" is the wavefront direction.

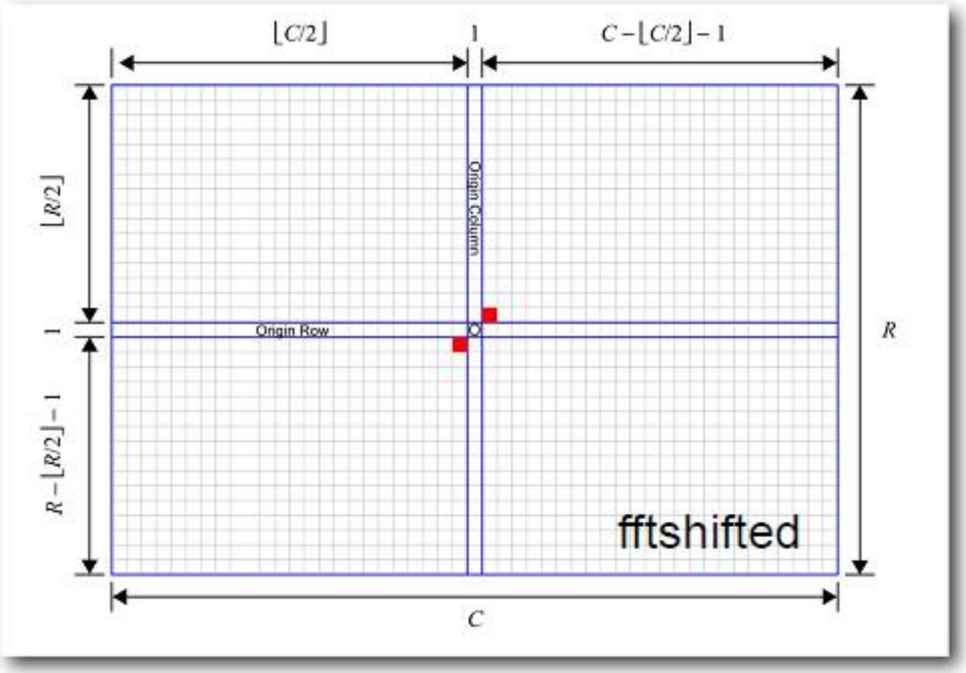


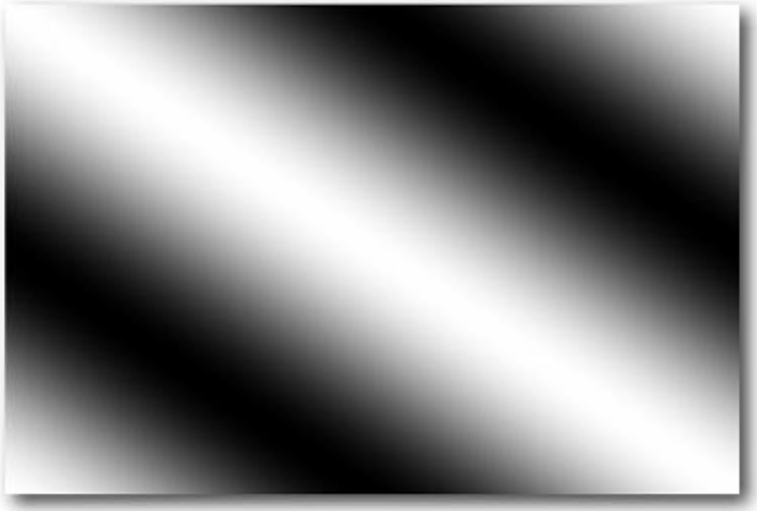


lowest-possible-frequency horizontal sinusoid "vertical" is the wavefront direction.

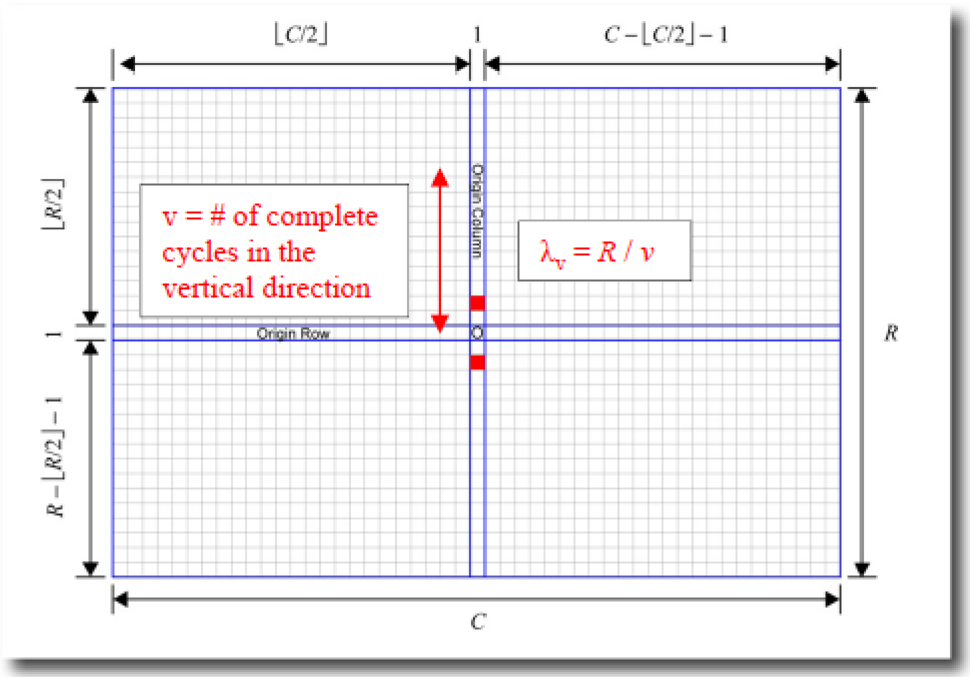


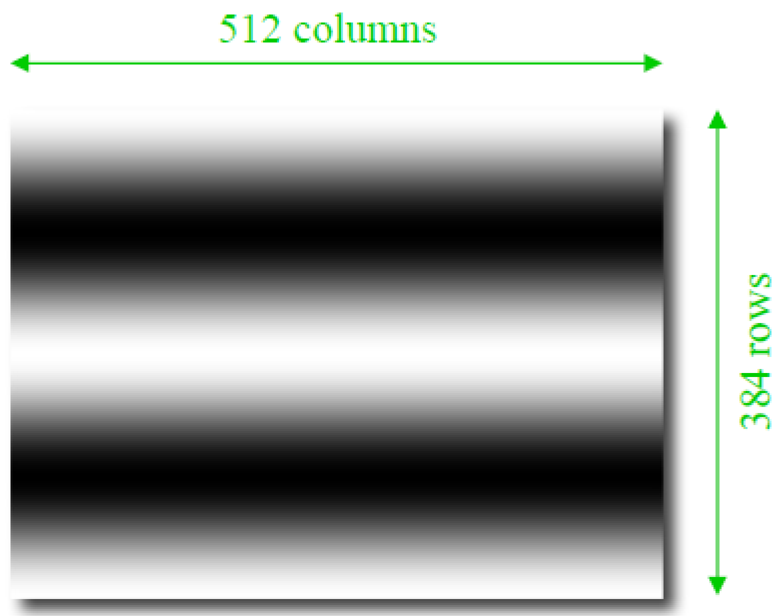
lowest-possible-frequency vertical sinusoid “negative diagonal” is the wavefront direction.



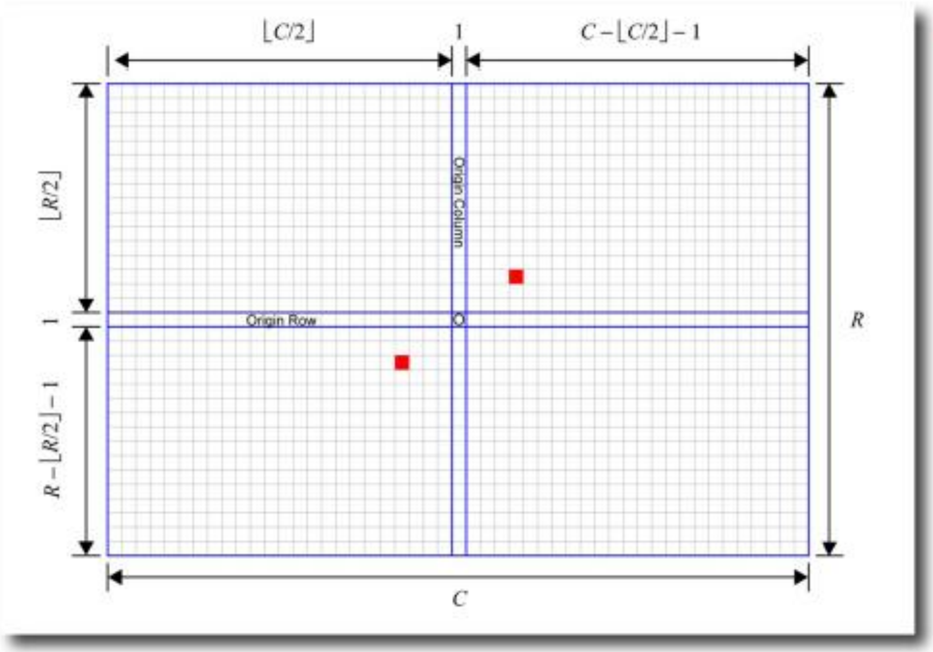


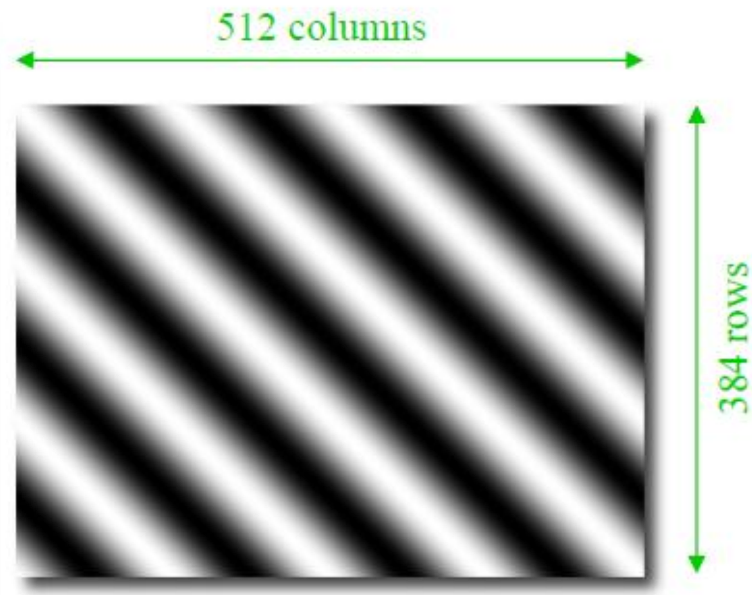
lowest-possible-frequency negative diagonal sinusoid





frequencies: $(u,v) = (0,2)$; wavelength: $\lambda_v = 192$





frequencies: $(u,v) = (4,3)$; wavelengths: $(\lambda_u, \lambda_v) = (128,128)$

Sinusoidové vzory sa zobrazia vo frekvenčnom spektre ako body.

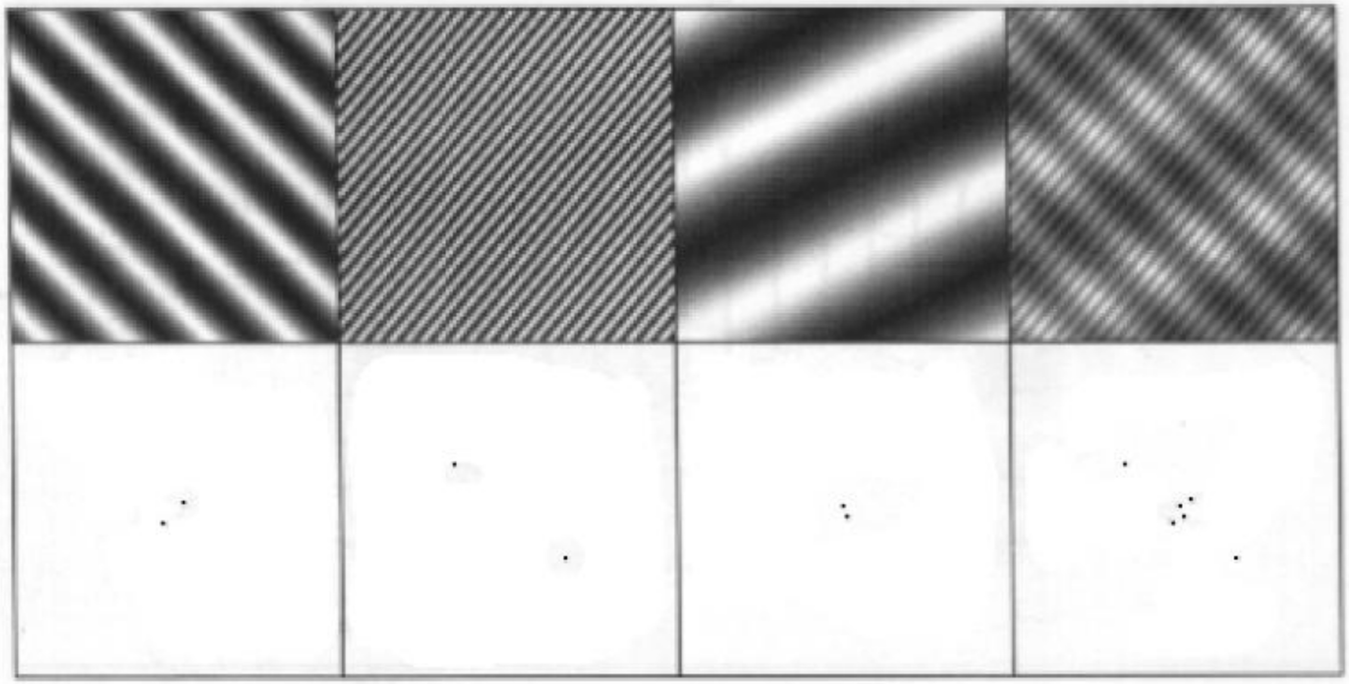


Figure 2: Images with perfectly sinusoidal variations in brightness: The first three images are represented by two dots. You can easily see that the position and orientation of those dots have something to do with what the original image looks like. The 4th image is the sum of the first three.
(Taken from p.177 of [1].)

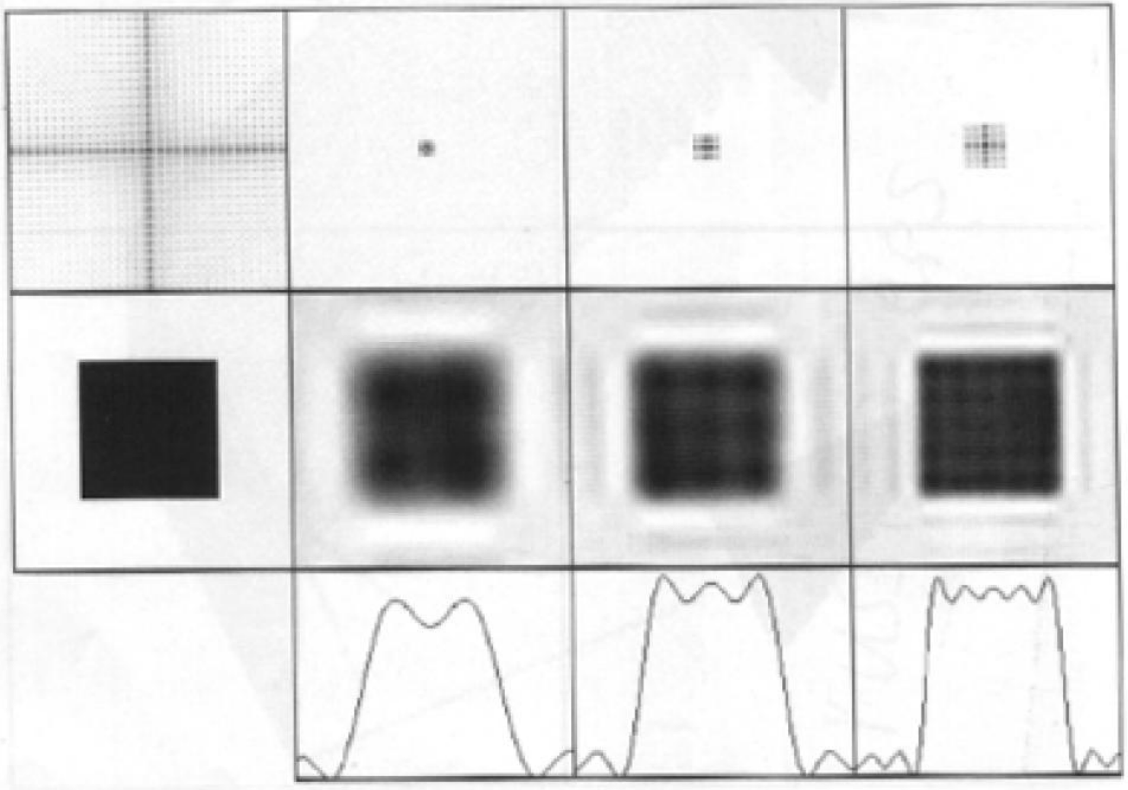


Figure 1: Images in the spatial domain are in the middle row, and their frequency space are shown on the top row. The bottom row shows the varying brightness of the horizontal line through the center of an image.
(Taken from p.178 of [1].)

- Nízke frekvencie sú pri strede a vysoké na okrajoch

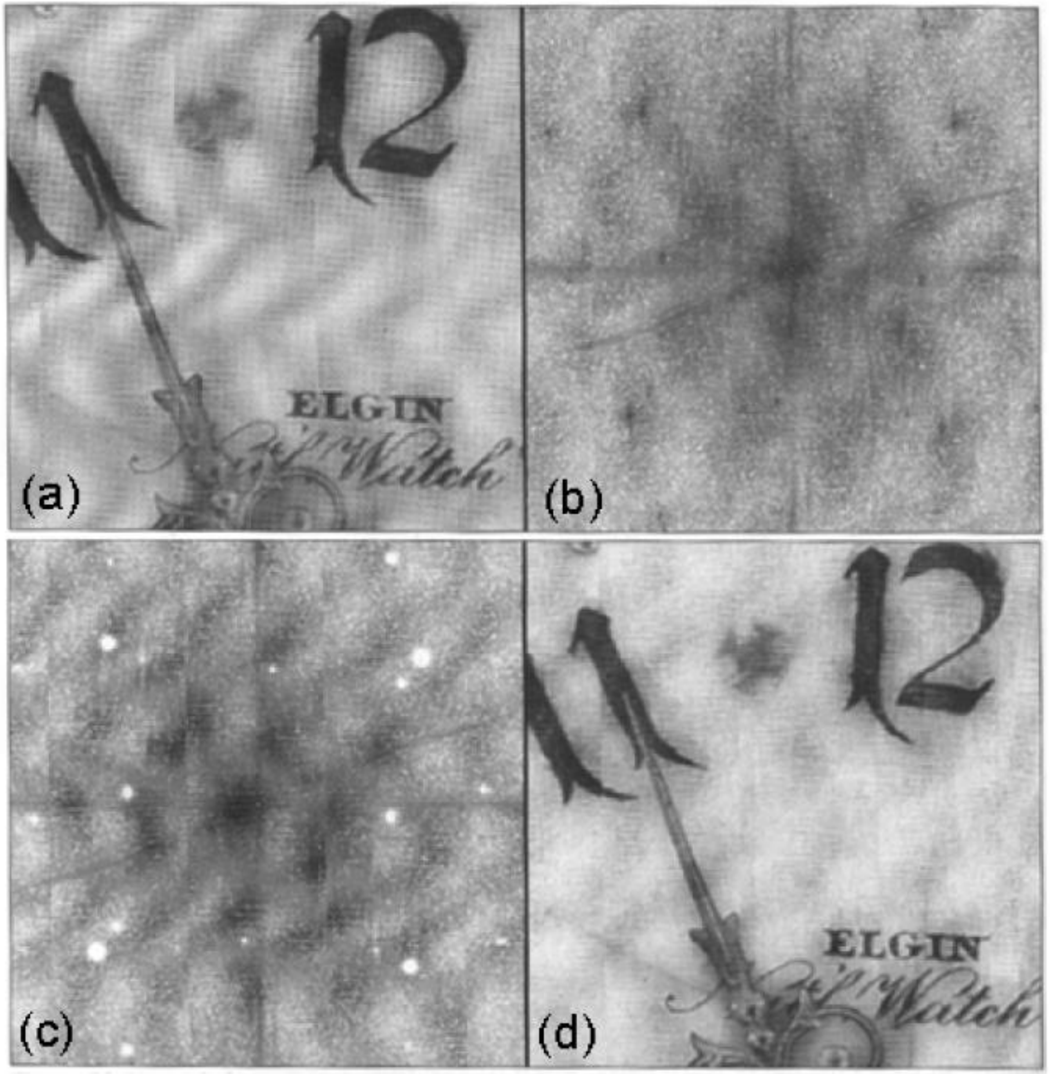
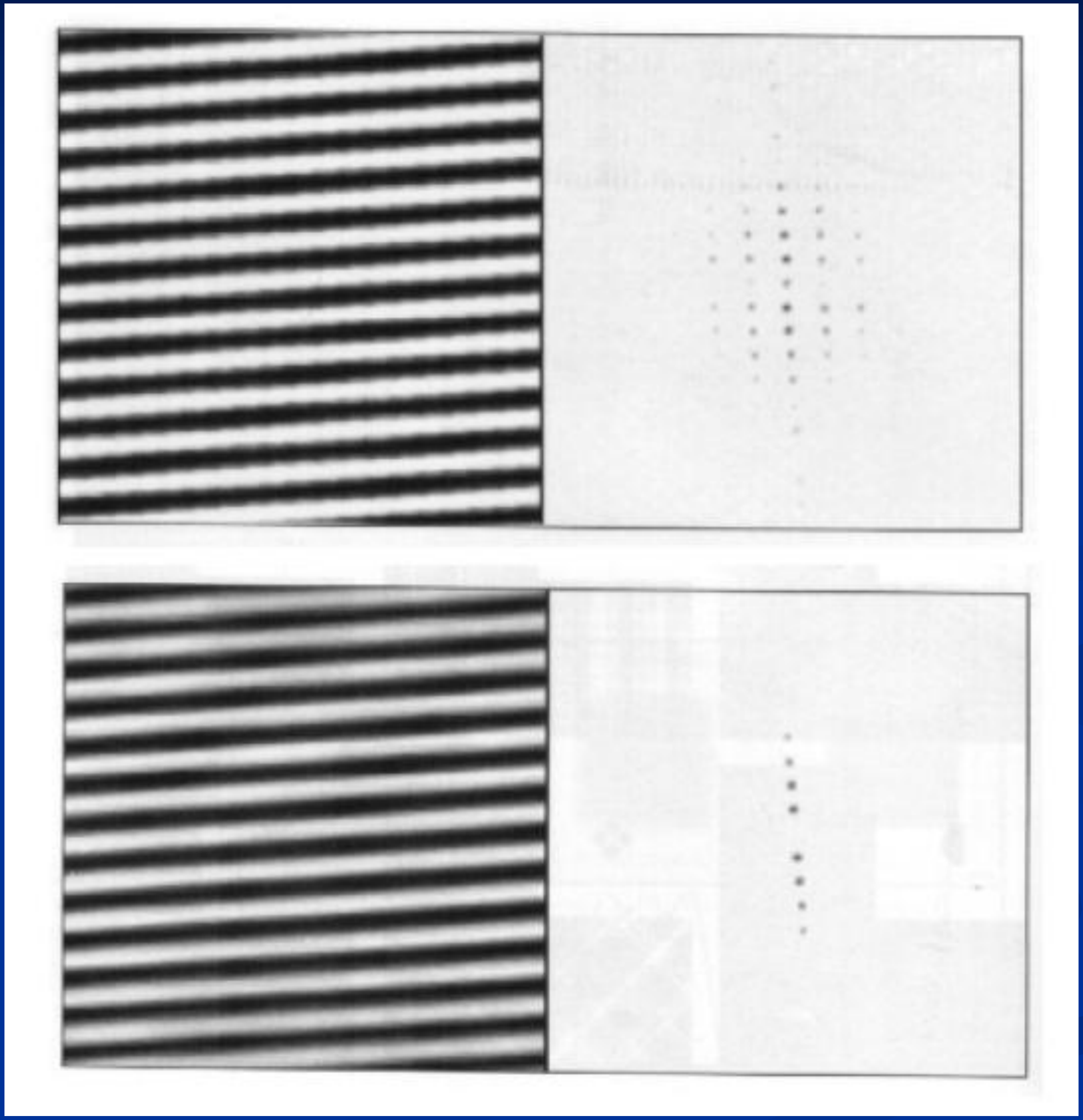
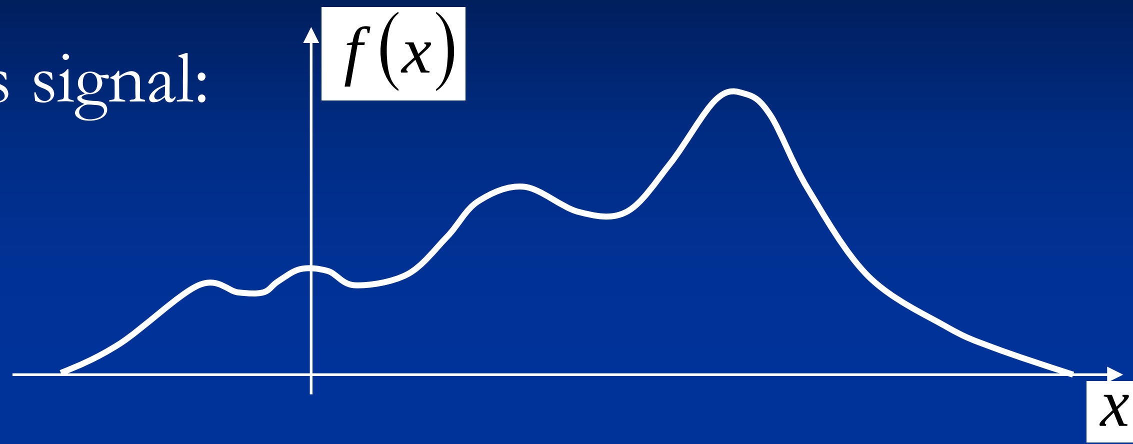


Figure 3: (a) Our dirty looking photocopied image. (b) The representation of our image in the frequency space, i.e. the star diagram. Look, you can see stars! (c) Those stars, however, do no good to the image, so we rub them out. (d) Reconstruct the image using (c) and those dirty spots on the original image are gone! (Images taken from p.204 of [1].)



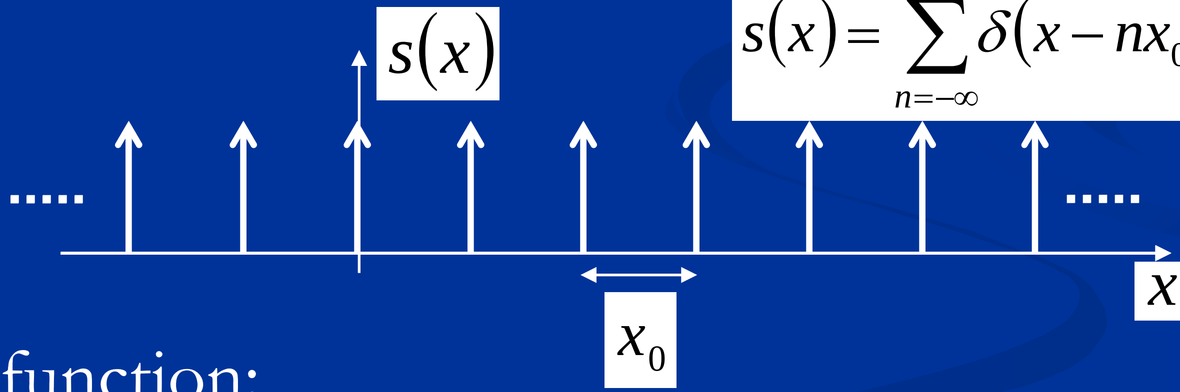
Sampling Theorem

Continuous signal:



Shah function (Impulse train):

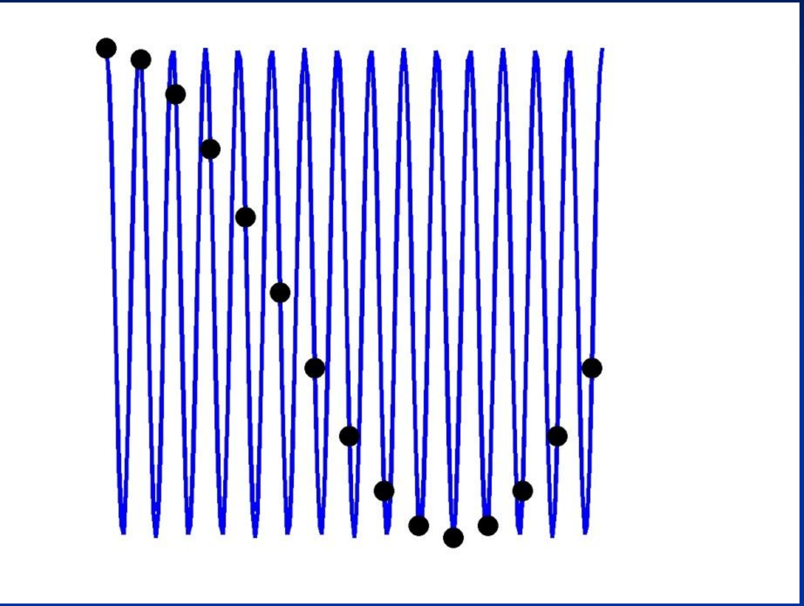
$$s(x) = \sum_{n=-\infty}^{\infty} \delta(x - nx_0)$$

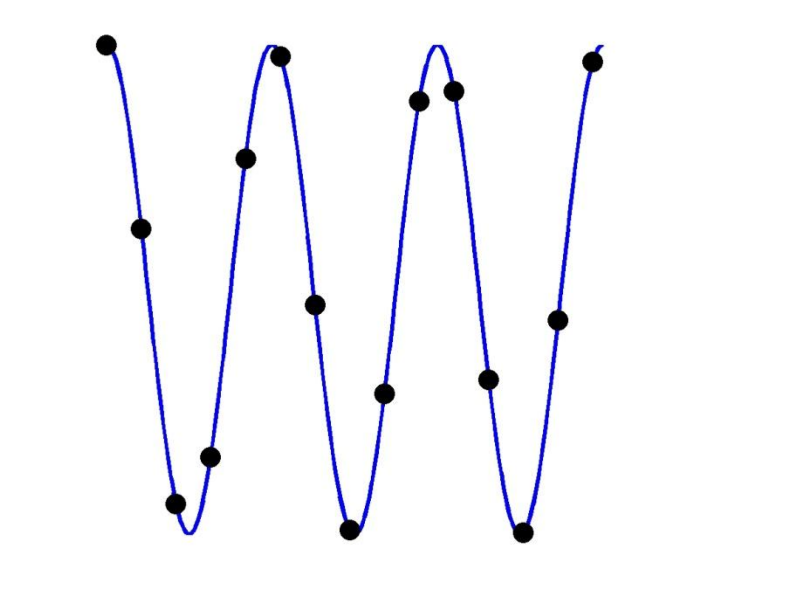


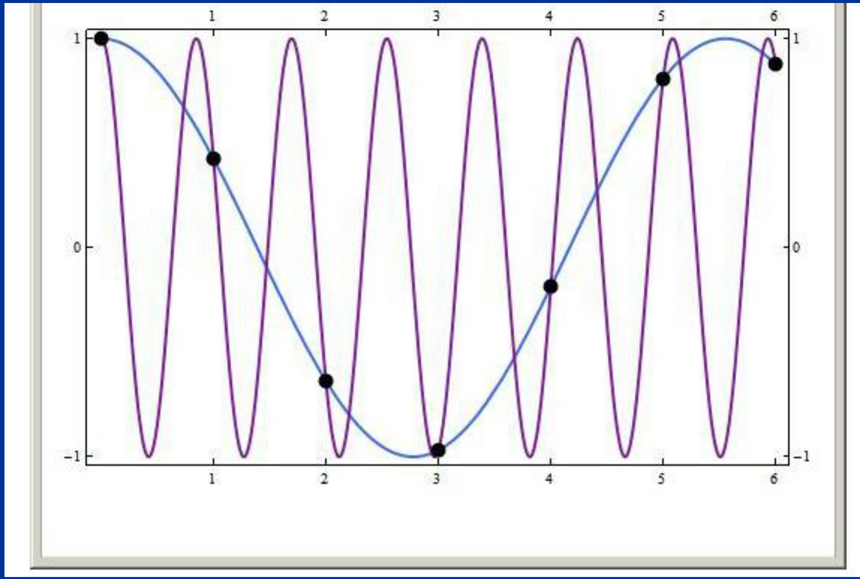
Sampled function:

$$f_s(x) = f(x)s(x) = f(x)\sum_{n=-\infty}^{\infty} \delta(x - nx_0)$$

Sampling







Sampling and the Nyquist rate

- **Pri vzorkovaní spojitej funkcie môže vzniknúť aliasing** ak vzorkovacia frekvencia nie je dostatočne vysoká
- Vzorkovacia frekvencia musí byť taká vysoká aby zachytila aj tie najvyššie frekvencie obrazu

Sampling and the Nyquist rate

- Predísť aliasingu:
- Vzorkovacia frekvencia > 2 * max frekvencia v obraze
  - Treba viac ako 2 vzorky na periódu
  - Minimálna vzorkovacia frekvencia sa nazýva **Nyquist rate**

Diskrétna Fourierova transformácia

$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-i2\pi ux/N}$$

for $u = 0, 1, 2, \ldots, N-1,$

$$f(x) = \sum_{u=0}^{N-1} F(u) e^{i2\pi ux/N}$$

for $x = 0, 1, 2, \ldots, N-1$

Rozšírenie DFT do 2D

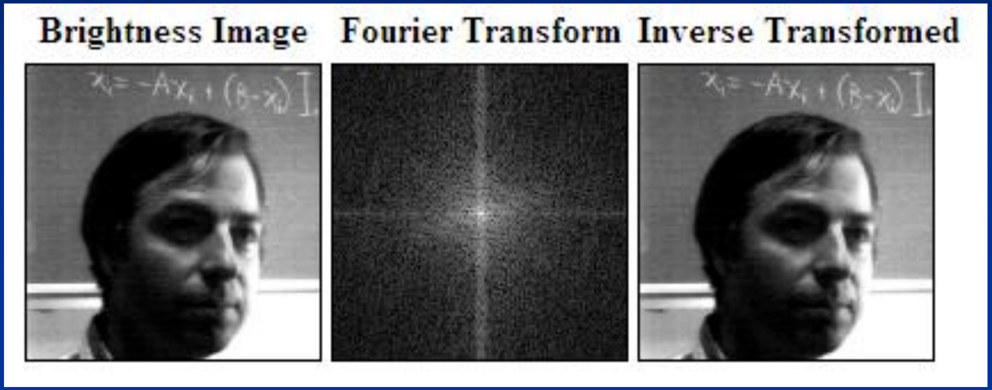
- Predpokladajme že f(x,y) je M x N.
- DFT $$F(u,v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})}$$

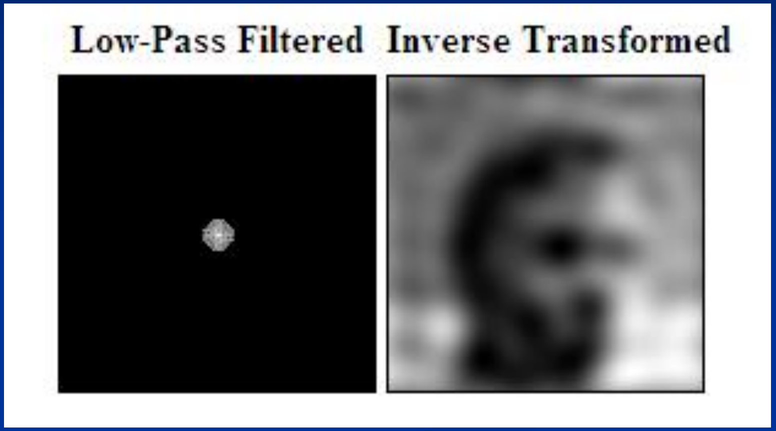
$$(u = 0, 1, \ldots, M-1,\ v = 0, 1, \ldots, N-1)$$

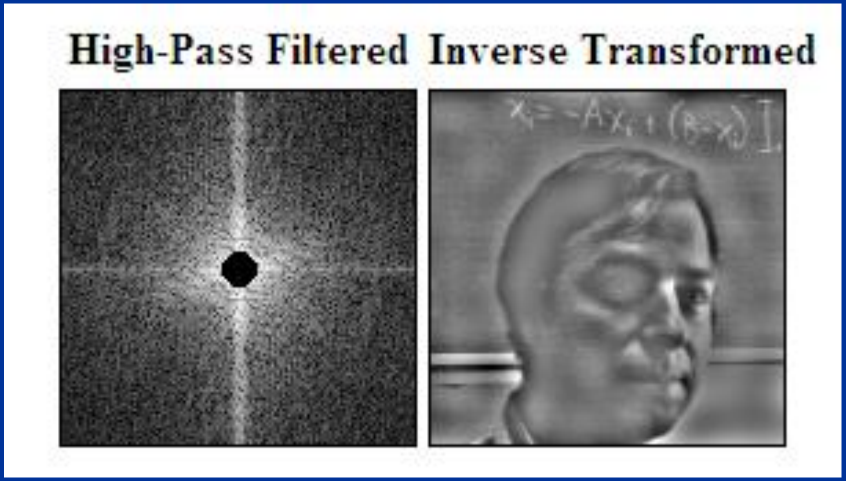
- Inverzná DFT: $$f(x,y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{j2\pi(\frac{ux}{M}+\frac{vy}{N})}$$

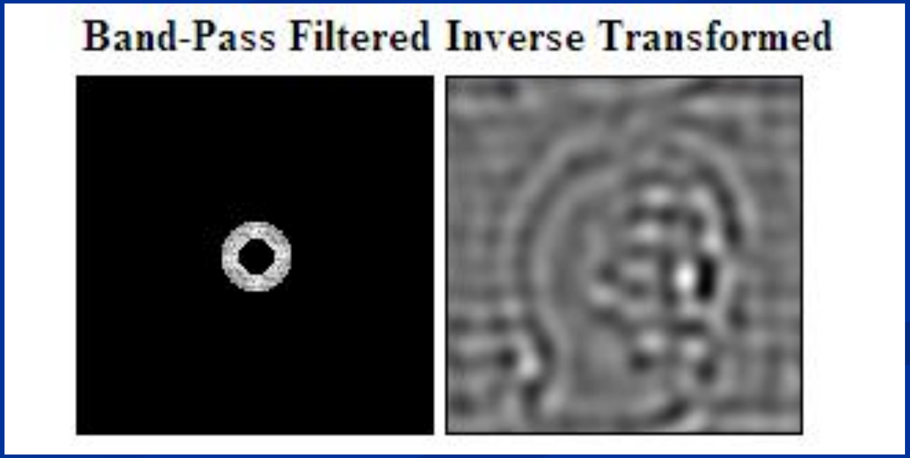
$$(x = 0, 1, \ldots, M-1,\ y = 0, 1, \ldots, N-1)$$

Filtre


Brightness Image
Fourier Transform
Inverse Transformed


Low-Pass Filtered
Inverse Transformed

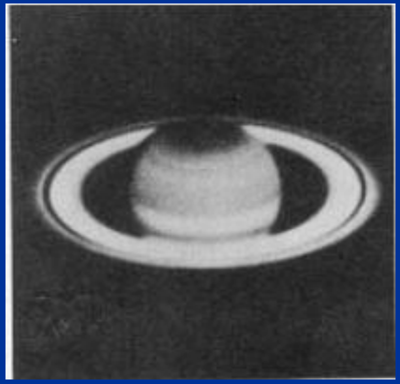

High-Pass Filtered
Inverse Transformed


Band-Pass Filtered
Inverse Transformed

Vizualizácia DFT

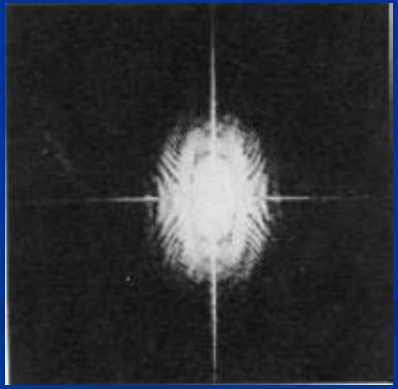
- Väčšinou zobrazujeme |F(u,v)|
- Dynamický rozsah |F(u,v)| je obvykle veľmi vysoký

- Aplikujeme logaritmus: $D(u,v) = c \; log(1 + |F(u,v)|)$
  (c je konštanta)



original image

before scaling



after scaling

DFT Properties: (1) Separability

- The 2D DFT can be computed using 1D transforms **only**:

Forward DFT:

$$F(u,v) = \frac{1}{N} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(\frac{ux+vy}{N})}$$

Inverse DFT:

$$f(x,y) = \frac{1}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} F(u,v) e^{j2\pi(\frac{ux+vy}{N})}$$

kernel is separable:

$$e^{-j2\pi(\frac{ux+vy}{N})} = e^{-j2\pi(\frac{ux}{N})} e^{-j2\pi(\frac{vy}{N})}$$

DFT Properties: (1) Separability

- Rewrite F(u,v) as follows:

$$F(u,v)=\frac{1}{N}\sum_{x=0}^{N-1} e^{-j2\pi(\frac{ux}{N})}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(\frac{vy}{N})}$$

- Let's set:

$$\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(\frac{vy}{N})}=F(x,v)$$

- Then:

$$F(u,v)=\frac{1}{N}\sum_{x=0}^{N-1} e^{-j2\pi(\frac{ux}{N})}F(x,v)$$

DFT Properties: (1) Separability

- How can we compute F(x,v)?

$$\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(\frac{vy}{N})} = F(x,v) = N\left(\frac{1}{N}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(\frac{vy}{N})}\right)$$

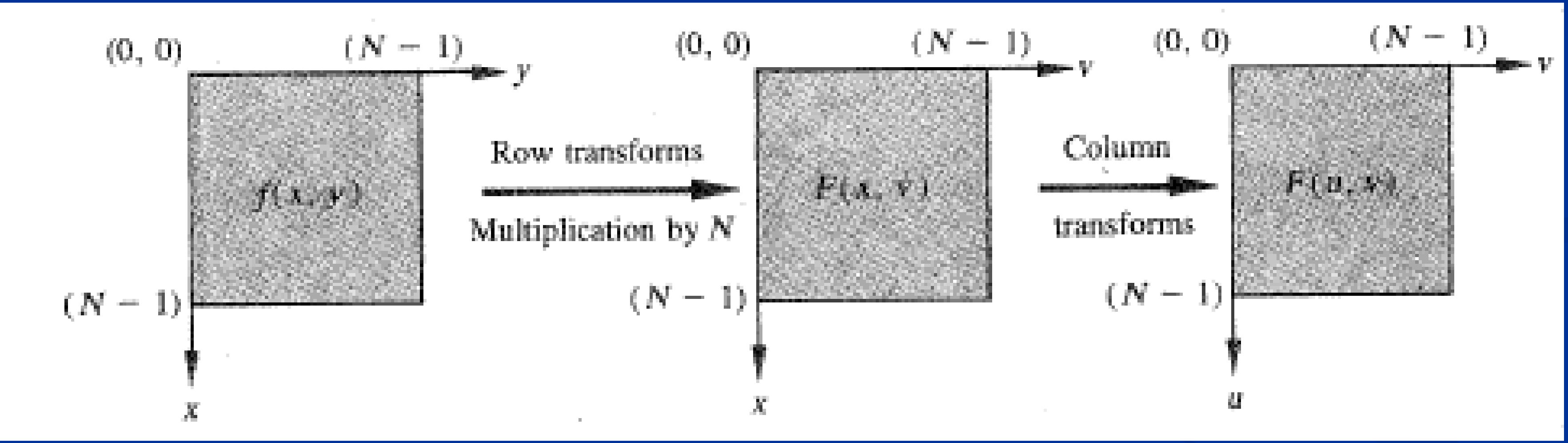
N x DFT of rows of f(x,y)

- How can we compute F(u,v)?

$$F(u,v) = \frac{1}{N}\sum_{x=0}^{N-1} e^{-j2\pi(\frac{ux}{N})} F(x,v)$$

DFT of cols of F(x,v)

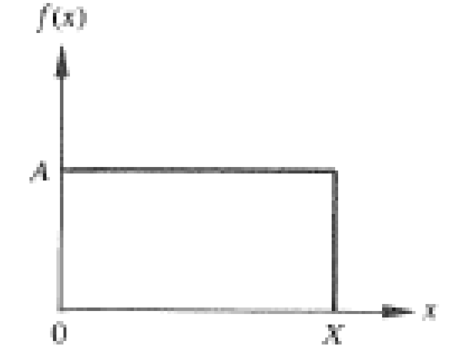
DFT Properties: (1) Separability



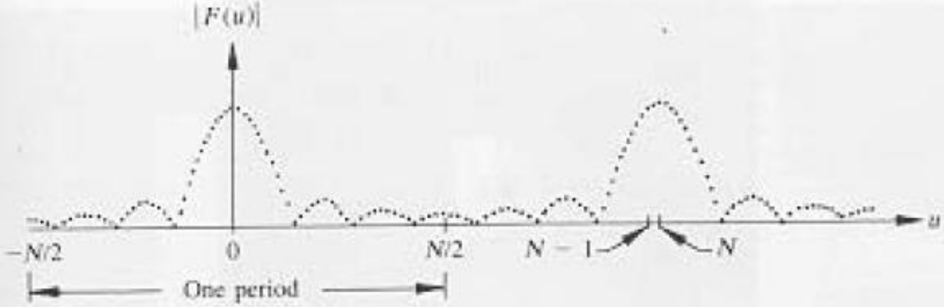
DFT Properties: (2) Periodicity

- The DFT and its inverse are periodic with period N

$$F(u,v) = F(u+N,v) = F(u,v+N) = F(u+N,v+N)$$



(a)



(a)

DFT Properties: (3) Symmetry

- If f(x,y) is real, then

$$F(u,v) = F^*(-u,-v) ==> |F(u,v)| = |F(-u,-v)|$$

$$f(x,y) \text{ real and even} \Leftrightarrow F(u,v) \text{ real and even}$$

$$f(x,y) \text{ real and odd} \Leftrightarrow F(u,v) \text{ imaginary and odd}$$

DFT Properties: (4) Translation

f(x,y) ⟷ F(u,v)

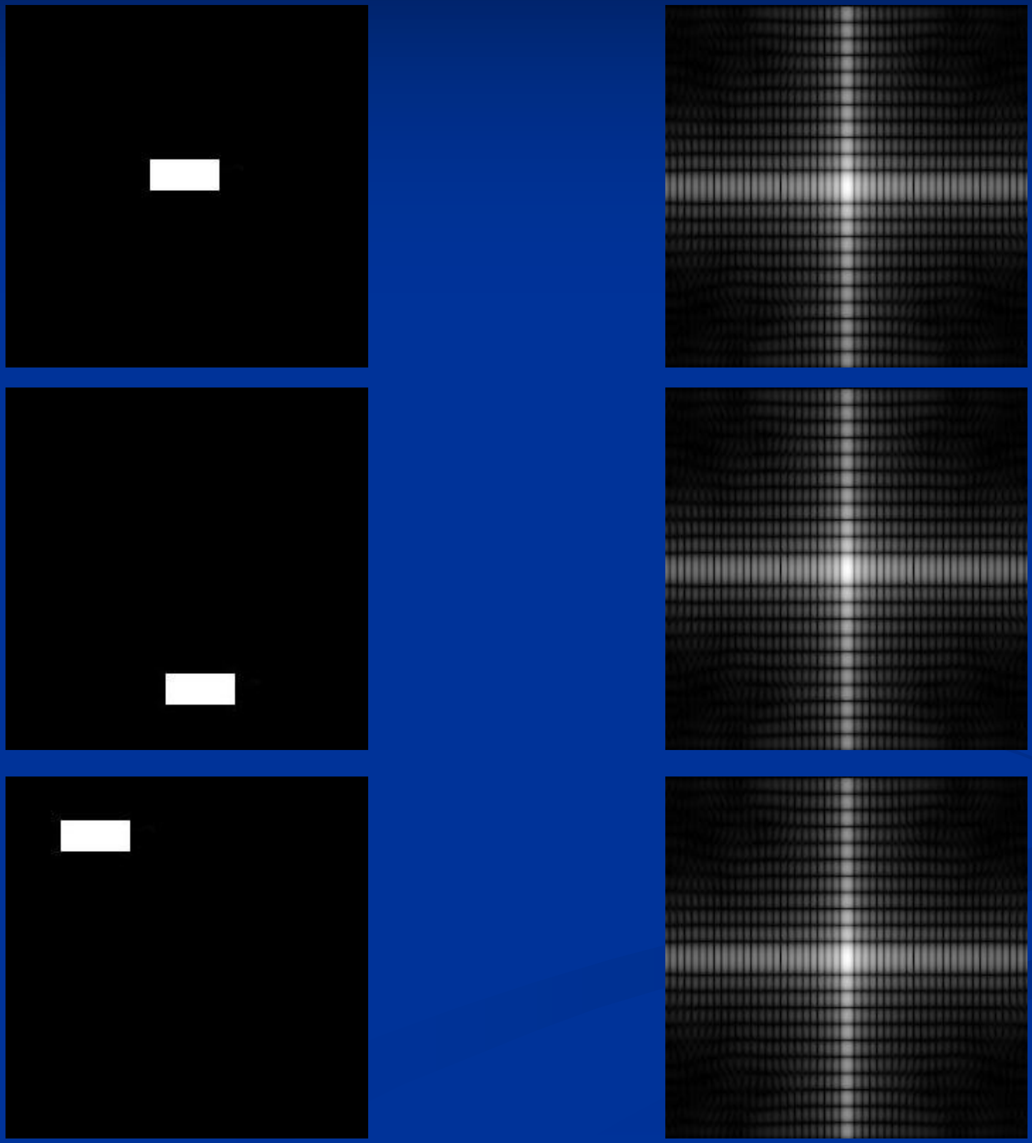
- Translation is spatial domain:

$$f(x-x_0, y-y_0) \Longleftrightarrow F(u,v)e^{-j2\pi(\frac{ux_0+vy_0}{N})}$$

- Translation is frequency domain:

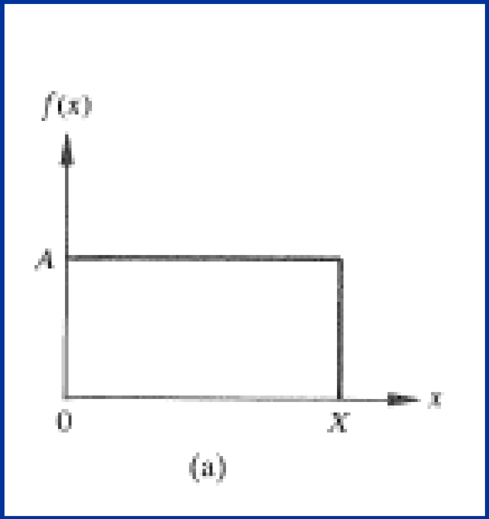
$$f(x,y)e^{j2\pi(\frac{u_0x+v_0y}{N})} \Longleftrightarrow F(u-u_0, v-v_0)$$

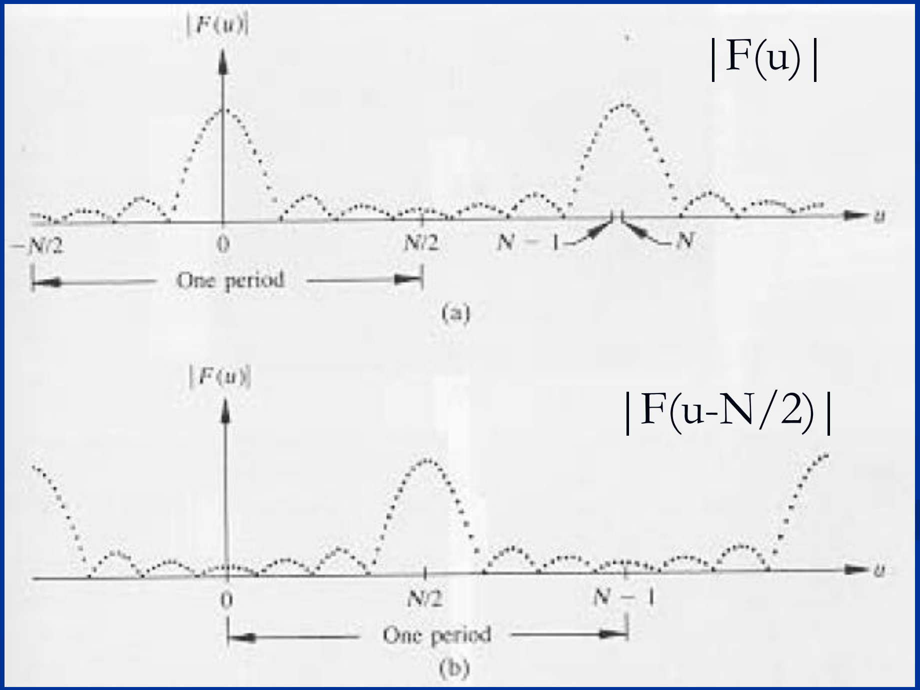
DFT Properties: (4) Translation



DFT Properties: (4) Translation

Warning: to show a full period, we need to translate the origin of the transform at **u=N/2** (or at **(N/2,N/2)** in 2D)





DFT Properties: (4) Translation

- To move F(u,v) at (N/2, N/2), take $u_0 = v_0 = N/2$

Using

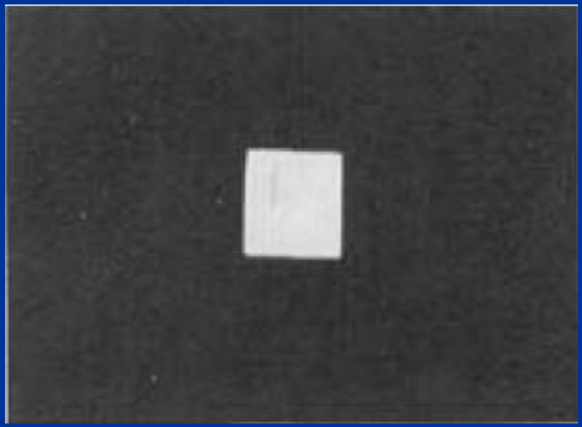
$$f(x,y)e^{j2\pi(\frac{u_0x+v_0y}{N})} \Leftrightarrow F(u-u_0, v-v_0)$$

$$e^{j2\pi(\frac{\frac{N}{2}x+\frac{N}{2}y}{N})} = e^{j\pi(x+y)} = (-1)^{x+y}$$

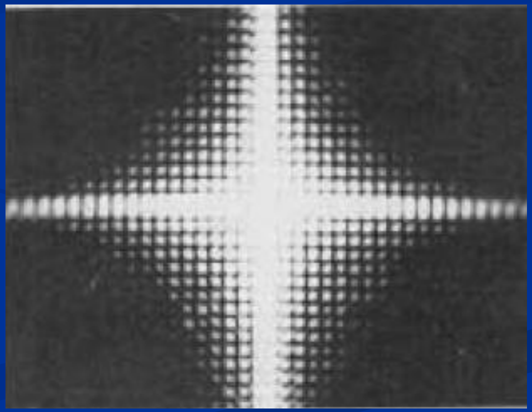
$$f(x,y)(-1)^{x+y} \Leftrightarrow F(u-N/2, v-N/2)$$

DFT Properties: (4) Translation

$$f(x,y)(-1)^{x+y} \Longleftrightarrow F(u-N/2,\, v-N/2)$$



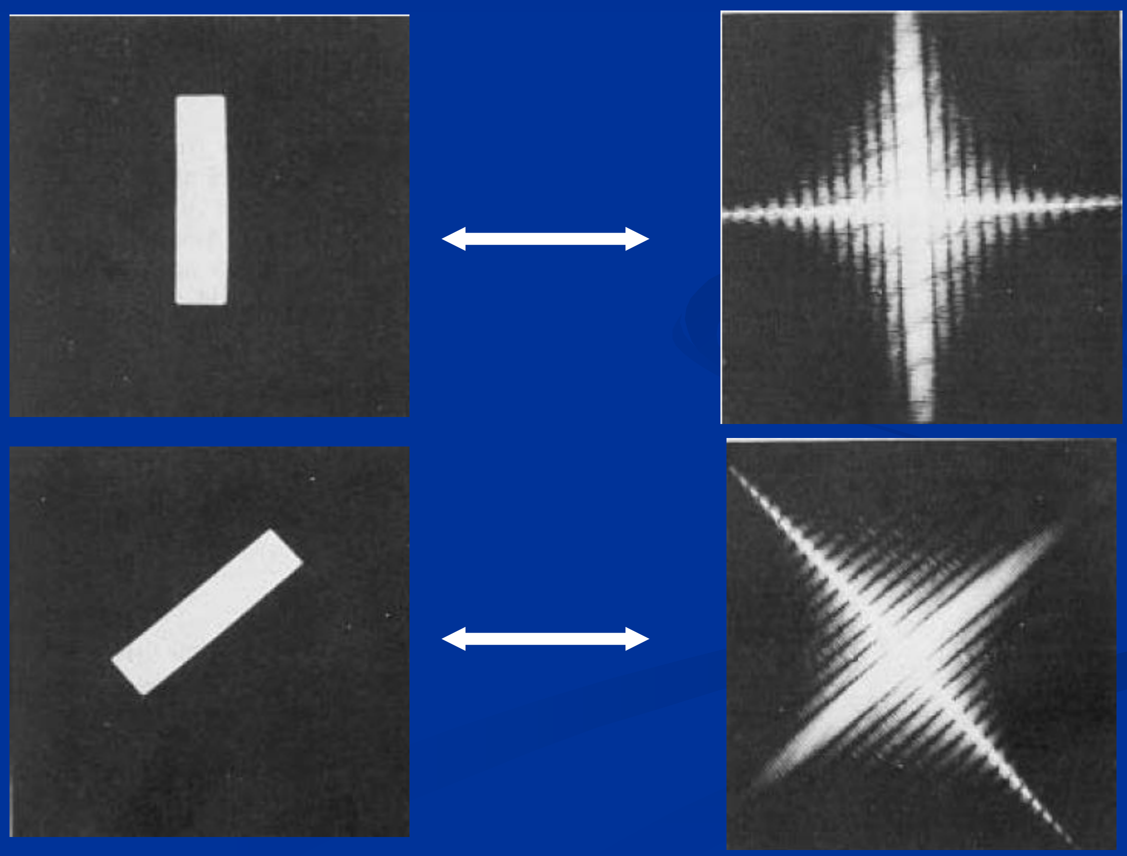
no translation



after translation

DFT Properties: (5) Rotation

- Otočením f(x,y) o uhol θ, sa otočí F(u,v) o ten istý uhol θ



DFT Properties: (6) Addition/Multiplication

$$F[f(x,y)+g(x,y)]=F[f(x,y)]+F[g(x,y)]$$

but …

$$F[f(x,y)g(x,y)]\neq F[f(x,y)]F[g(x,y)]$$

DFT Properties: (7) Scale

$$af(x,y) \Longleftrightarrow aF(u,v)$$

DFT Properties: (8) Average value

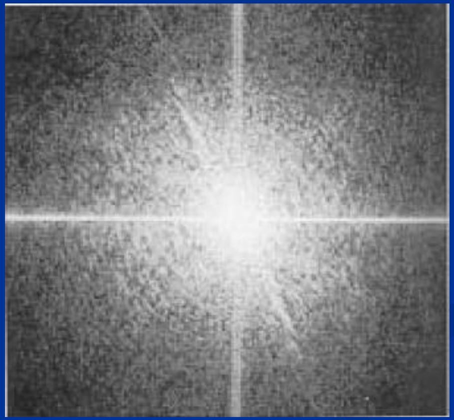
Average: $$\bar{f}(x,y) = \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)$$

F(u,v) at u=0, v=0: $$F(0,0) = \frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)$$

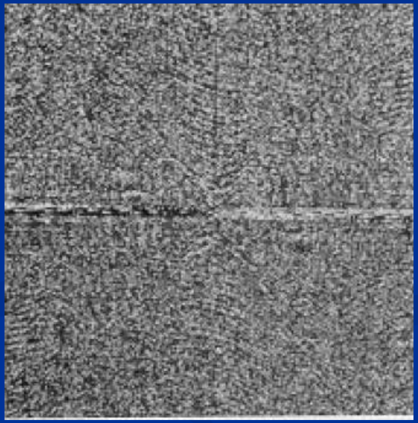
So: $$\bar{f}(x,y) = \frac{1}{N}F(0,0)$$

Magnitude and Phase of DFT

- What is more important?



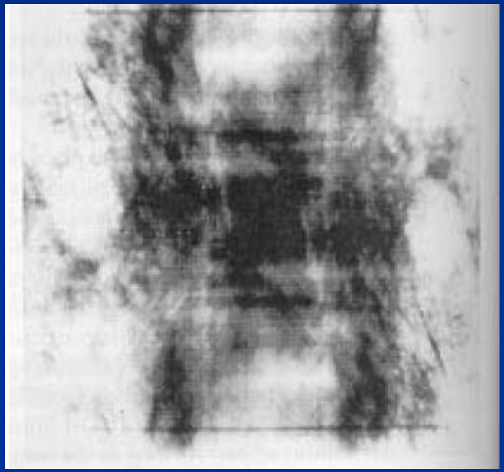
magnitude



phase

- **Hint:** use inverse DFT to reconstruct the image using magnitude or phase only information

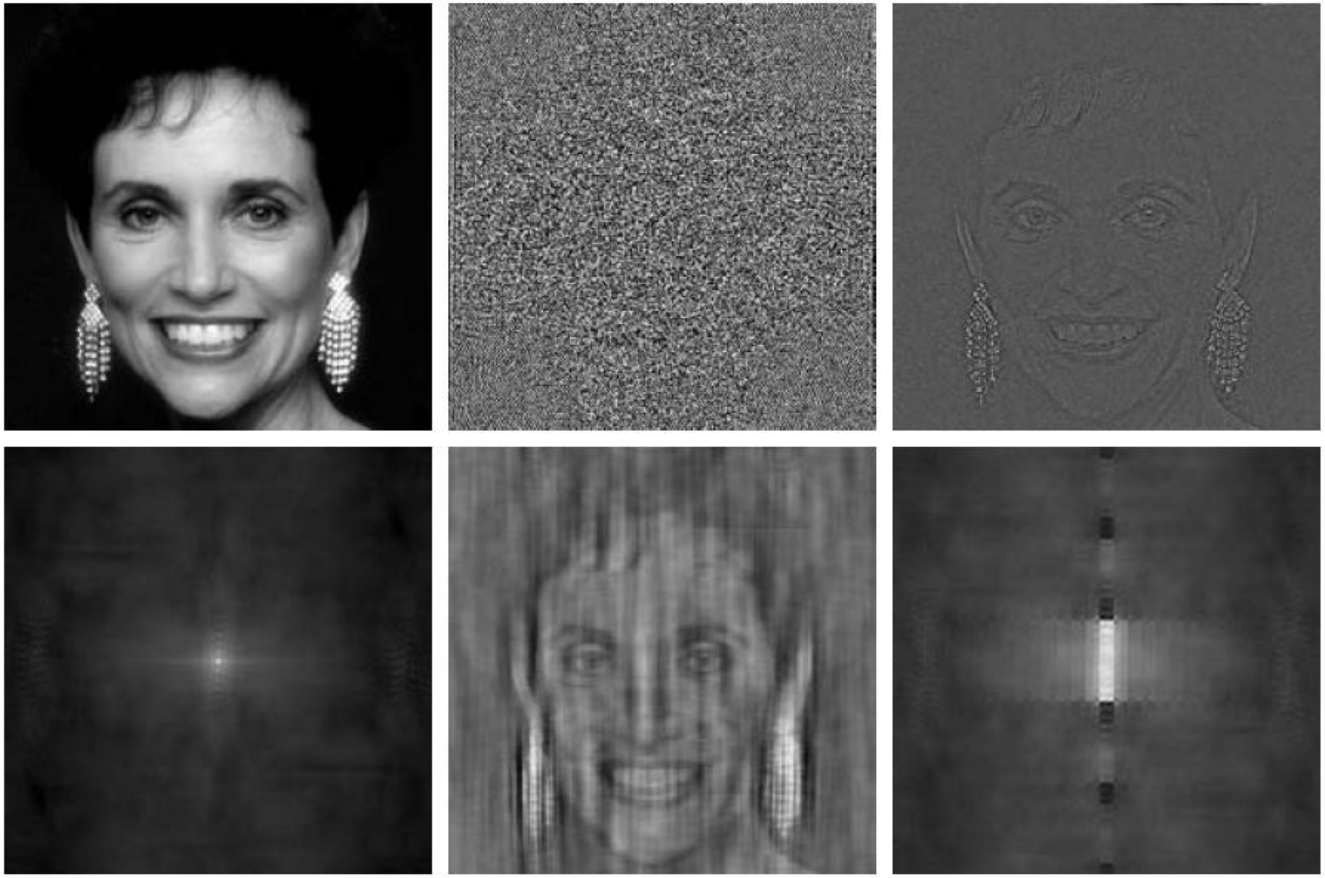
Magnitude and Phase of DFT



Reconstructed image using
magnitude only
(i.e., magnitude determines the
contribution of each component!)

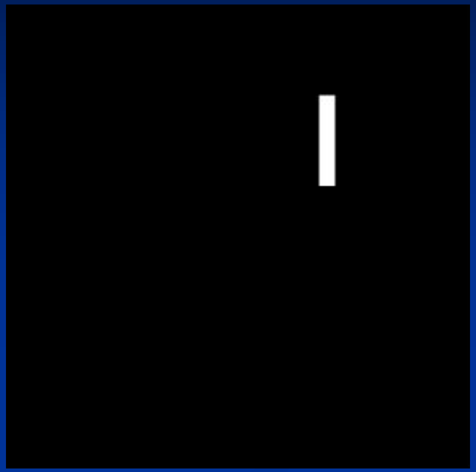
Reconstructed image using
phase only
(i.e., phase determines
which components are present!)

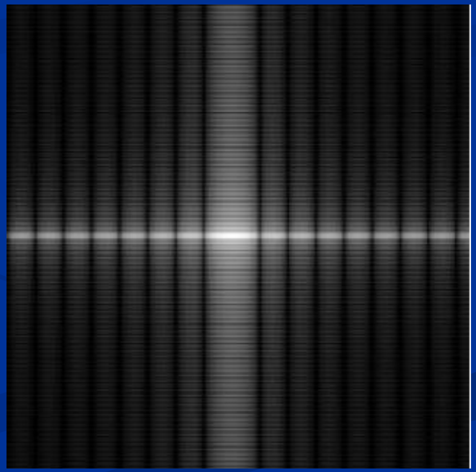
Magnitude and Phase of DFT



a b c
d e f

FIGURE 4.27 (a) Woman. (b) Phase angle. (c) Woman reconstructed using only the phase angle. (d) Woman reconstructed using only the spectrum. (e) Reconstruction using the phase angle corresponding to the woman and the spectrum corresponding to the rectangle in Fig. 4.24(a). (f) Reconstruction using the phase of the rectangle and the spectrum of the woman.





Výpočtová náročnosť DFT

$$F_n = \sum_{k=0}^{N-1} f_k e^{-i\frac{2\pi}{N}kn}$$

discrete Fourier transform — DFT

$\{f_0, f_1, \ldots, f_{N-1}\}$ – vstupné body
$\{F_0, F_1, \ldots, F_{N-1}\}$ – výsledok Fourierovej transformácie

Výpočet pomocou 2 cyklov – > zložitosť $O(N^2)$

Snaha zredukovať na $O(N\log_2 N)$ – > FFT

Fast Fourier Transform - FFT

$$F_n = \sum_{k=0}^{N-1} f_k e^{-i\frac{2\pi}{N}kn}$$

Pre N=8

$$F_n = f_0 + f_1 e^{-i\frac{2\pi}{8}n} + f_2 e^{-i\frac{2\pi}{8}2n} + f_3 e^{-i\frac{2\pi}{8}3n} + f_4 e^{-i\frac{2\pi}{8}4n} + f_5 e^{-i\frac{2\pi}{8}5n} + f_6 e^{-i\frac{2\pi}{8}6n} + f_7 e^{-i\frac{2\pi}{8}7n}$$

Rozdelíme na párne a nepárne členy

$$F_n = \left[f_0 + f_2 e^{-i\frac{2\pi}{8}2n} + f_4 e^{-i\frac{2\pi}{8}4n} + f_6 e^{-i\frac{2\pi}{8}6n} \right] + e^{-i\frac{2\pi}{8}n} \left[f_1 + f_3 e^{-i\frac{2\pi}{8}2n} + f_5 e^{-i\frac{2\pi}{8}4n} + f_7 e^{-i\frac{2\pi}{8}6n} \right]$$

Urobíme to isté vo zátvorkách

$$F_n = \left[\left(f_0 + f_4 e^{-i\frac{2\pi}{8}4n} \right) + e^{-i\frac{2\pi}{8}2n} \left(f_2 + f_6 e^{-i\frac{2\pi}{8}4n} \right) \right] + e^{-i\frac{2\pi}{8}n} \left[\left(f_1 + f_5 e^{-i\frac{2\pi}{8}4n} \right) + e^{-i\frac{2\pi}{8}2n} \left(f_3 + f_7 e^{-i\frac{2\pi}{8}4n} \right) \right]$$

$$F_n = \left[\left(f_0 + f_4 e^{-i\frac{2\pi}{8}4n} \right) + e^{-i\frac{2\pi}{8}2n} \left(f_2 + f_6 e^{-i\frac{2\pi}{8}4n} \right) \right] + e^{-i\frac{2\pi}{8}n} \left[\left(f_1 + f_5 e^{-i\frac{2\pi}{8}4n} \right) + e^{-i\frac{2\pi}{8}2n} \left(f_3 + f_7 e^{-i\frac{2\pi}{8}4n} \right) \right]$$

$$F_n = \left[\left(f_0 + f_4 e^{-i\pi n} \right) + e^{-i\frac{\pi}{2}n} \left(f_2 + f_6 e^{-i\pi n} \right) \right] + e^{-i\frac{\pi}{4}n} \left[\left(f_1 + f_5 e^{-i\pi n} \right) + e^{-i\frac{\pi}{2}n} \left(f_3 + f_7 e^{-i\pi n} \right) \right]$$

$$e^{i(\phi + 2\pi)} = e^{i\phi}$$

Suma vo vnútorných zátvorkách je rovnaká pre n=0, 2, 4, 6 a pre n=1, 3, 5, 7

Takže máme 4 (zátvorky) x 2 súčtov na najnižšej úrovni

Keďže n=1, 3, 5, 7 zodpovedá polovici periódy Π a platí $e^{i(\phi + \pi)} = -e^{i\phi}$

dostaneme 1 pre n=0, 2, 4, 6 a

- 1 pre n=1, 3, 5, 7

$$F_n = \left[\left(f_0 + f_4 e^{-i\pi n}\right) + e^{-i\frac{\pi}{2}n}\left(f_2 + f_6 e^{-i\pi n}\right)\right] + e^{-i\frac{\pi}{4}n}\left[\left(f_1 + f_5 e^{-i\pi n}\right) + e^{-i\frac{\pi}{2}n}\left(f_3 + f_7 e^{-i\pi n}\right)\right]$$

V hranatých zátvorkách je perióda exponentu n=4
– > n=0, 4;   n=1, 5;   n=2, 6 a   n=3, 7.

Pre n=0, 4 factor je 1
pre n=2, 6 je to -1;
pre n=1, 5 je -i
a pre n=3, 7 je i.

Takže máme 2 (zátvorky) x 4 súčtov na strednej úrovni

$$F_n = \left[\left(f_0 + f_4 e^{-i\pi n} \right) + e^{-i\frac{\pi}{2}n} \left(f_2 + f_6 e^{-i\pi n} \right) \right] + e^{-i\frac{\pi}{4}n} \left[\left(f_1 + f_5 e^{-i\pi n} \right) + e^{-i\frac{\pi}{2}n} \left(f_3 + f_7 e^{-i\pi n} \right) \right]$$

Na najvyššej úrovni máme 1 sumu a periódu exponentu n=8
– > 1 x 8 súčtov
Polovicu z nich môžeme vypočítať iba zmenou znamienka

Na každej úrovni urobíme 8 operácií (n)
Máme 3 = log 8 úrovne (log n)

} O(n log n)

FFT - algoritmus

$$\{f_0, f_1, f_2, f_3, f_4, f_5, f_6, f_7\} \rightarrow \{f_0, f_4, f_2, f_6, f_1, f_5, f_3, f_7\}.$$

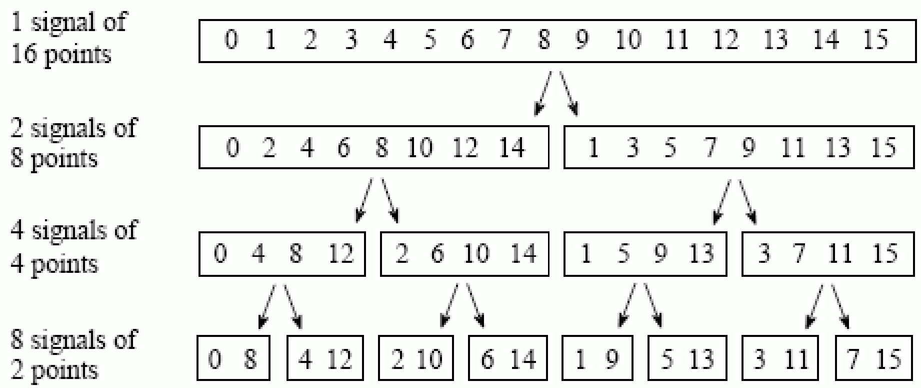
Prepare input data for summation — put them into convenient order;
For every summation level:
    For every exponent factor of the half-period:
        Calculate factor;
        For every sum of this factor:
            Calculate product of the factor and the second term of the sum;
            Calculate sum;

Poradie koeficientov pre FFT

$$F_n = \left[\left(f_0 + f_4 e^{-i\pi n}\right) + e^{-i\frac{\pi}{2}n}\left(f_2 + f_6 e^{-i\pi n}\right)\right] + e^{-i\frac{\pi}{4}n}\left[\left(f_1 + f_5 e^{-i\pi n}\right) + e^{-i\frac{\pi}{2}n}\left(f_3 + f_7 e^{-i\pi n}\right)\right]$$

0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7

Príklad pre N=16



Sample numbers in normal order		→	Sample numbers after bit reversal	
Decimal	*Binary*		*Decimal*	*Binary*
0	0000		0	0000
1	0001		8	1000
2	0010		4	0100
3	0011		12	1100
4	0100		2	0010
5	0101		10	1010
6	0110		6	0100
7	0111		14	1110
8	1000		1	0001
9	1001		9	1001
10	1010		5	0101
11	1011		13	1101
12	1100		3	0011
13	1101		11	1011
14	1110		7	0111
15	1111		15	1111