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Recursive subdivision of polygonal complexes and its applications in computer-aided geometric design

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Abstract

A method for subdividing polygonal complexes and identifying conditions to control their limit curves is presented. A polygonal complex is a sequence of panels where every two adjacent panels share one edge only. We formulate this problem and establish a general theory which has a number of applications in CAGD such as the generation of subdivision surfaces through predefined arbitrary network of curves. This is a further extension of the capability of these surfaces making them more attractive and more practical in surface modeling and computer graphics. One of the main advantages of the proposed scheme is that the regions of the surface between the interpolated curves do not have to be rectangular—a limitation of existing tensor-product based CAD systems. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Recursive subdivision; Curve interpolation; B-spline; Arbitrary curve networks; N-sided regions

1. Introduction

Recursive subdivision has been receiving extensive attention over the past few years in free-form surface modeling, multiresolution, and computer graphics (Lounsbery et al., 1997; Loop and DeRose, 1990; Peters and Nasri, 1997; Kobbelt, 1996; Reif, 1995; Dyn and Levin, 1990; Halstead et al, 1993; Peters, 1993; Prautzsch, 1995; Subdivision Methods for Geometric Design, 1995; Zorin et al., 1996). If provides definition of surfaces over arbitrary topology with many interpolation capabilities, and various robust algorithms for the interrogation of such surfaces. Recently, it has been used in character animation such as the short movie Geri’s game produced by Pixar (1998). A non-uniform subdivision

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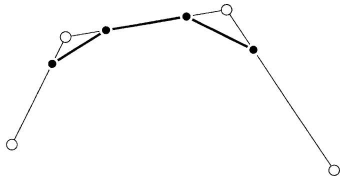

Fig. 1. A control polygon (hollow vertices) and its first Chaikin's subdivision (solid vertices).

scheme was also recently suggested by Sederberg et al. (1998). Subdivision schemes are is continuing to be more and more attractive but further extensions of capabilities still need to be developed. The proposed scheme of this paper can be used in this direction.

Basically, a recursive subdivision surface is the limit of a subdivision process in which an initial configuration P_0, often referred to as polyhedron, describing a surface is repeatedly refined. The configuration consists of a set of vertices, edges and faces which need not be planar. A set of rules is then applied to the configuration P_0 to generate another P_1 with more vertices and smaller faces. The process is recursively repeated and at the limit the configuration converges to a G^1 surface S. The subdivision schemes differ by the rules used to generate the new vertices. The Doo–Sabin approach (1978), for instance, is an extension of Chaikin's method to generate a smooth curve by repeated subdivision of a given control polygon. In Chaikin's subdivision of a polygon cp_i having vertices $(v_i)_{1\leqslant i\leqslant n}$, each edge e_i joining the two vertices v_{i-1} and v_i will generate an edge $\widehat{e_i}$, called *E-edge*, joining the vertices v_{i-1}^1 and v_i^0 of cp_{i+1} as illustrated in Fig. 1. The v_i^j are given by the following:

$$v_{i-1}^1 = \frac{3}{4}v_{i-1} + \frac{1}{4}v_i, \tag{1.1}$$

$$v_i^0 = \frac{3}{4}v_i + \frac{1}{4}v_{i-1}. \tag{1.2}$$

Note that in order to interpolate the endpoints v_1 and v_n, the first and last edges, called *end-legs*, are symmetrically extended about v_1 and v_n, respectively, prior to subdivision. Furthermore, each vertex v_i, except the endpoints, of cp_i will correspond to an edge, called *V-edge*, joining the two vertices v_i^0 and v_i^1 see Fig. 1. It was shown that the sequence of generated polygons will, at the limit, converge to a quadratic B-spline curves (Riesenfeld, 1975).

Based on this curve algorithm, the Doo–Sabin approach generates biquadratic tensor product B-spline surfaces. In the subdivision process of the polyhedron P_i, the new vertices of P_{i+1} are linear combinations of the vertices of P_i. The following rules apply to generate the new vertices $\widehat{v_i}$, on a face F having the vertices $(v_i)_{1\leqslant i\leqslant n}$:

$$\widehat{v_i} = \sum_{j=1}^{n} \alpha_{ij} v_j, \tag{1.3}$$

where the α_{ij}'s are given by:

$$\alpha_{ii} = \frac{n+5}{4n}, \tag{1.4}$$

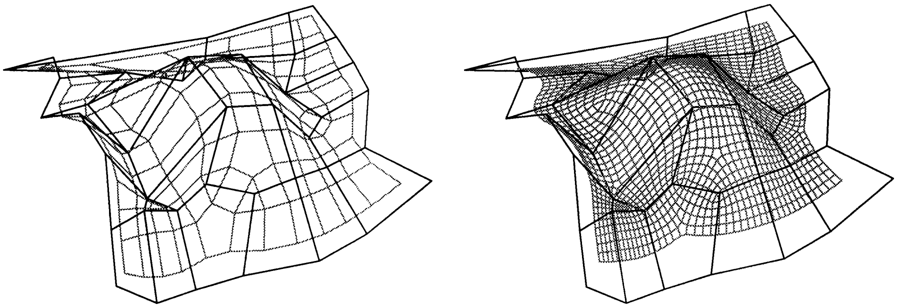

Fig. 2. The Doo–Sabin Approach: a polyhedron and its first subdivision (left), and its third subdivision (right) are shown.

$$\alpha_{ij} = \frac{3 + 2\cos\frac{2\pi(i-j)}{n}}{4n}. \tag{1.5}$$

P_{i+1} is constructed by linking the new vertices and generating three types of faces, as indicated in Fig. 2. Two types of polyhedra are known: open and closed. In the latter all vertices are called interior vertices whereas in the former the vertices can be interior or boundary[2] vertices, as illustrated in Fig 2. For each face F of P_i an F-face is constructed from the images of the vertices of F. Furthermore, each *non-boundary* edge e_r of P_i, that is an edge with at least one interior vertex, will correspond to an E-face that links the images of its two endpoints on its two common faces. Finally, for each interior vertex V of P_i a V-face is made by linking its images on the faces sharing V.

A cubic approach was also devised by Catmull and Clark (1978) which generate surfaces that are in general G^2 except at the *irregular* points, these are points that correspond to n-valent ($n \neq 4$) vertices of the surface.

Initially, the techniques were not practical and this has led to various extensions such as the capabilities of interpolating point, normals, and more recently curves. Several approaches to interpolate isolated open or closed curves by Recursive subdivision surfaces was recently devised (Nasri, 1997a, 1997b; Hoppee et al., 1994; Schweitzer, 1996). The problem can be stated as follows. Given a set of curve (c_i) defined by a set of tagged control polygons (cp_i), on a given polyhedron P describing a surface S. The latter can be forced to interpolate the B-spline curves of (cp_i). One approach to solve this problem (Nasri, 1997a, 1997b) was to modify the E- and V-faces generated from the edges and vertices of each control polygon cp_i, such that their subsequent subdivisions will result in a subdivision of cp_i at the same time. At the limit, the curve c_i of cp_i is interpolated by the limit surface of P. This technique ensures C^1 continuity across the interpolated quadratic curves. In (Hoppee et al., 1994), curves are interpolated by modifying the subdivision rules giving C^0 continuity across the interpolated cubic curve which is called *crease*. In general, changing the rules of subdivision may change the eigenvalues and eigenvectors of the subdivision matrix and consequently, the limit surface needs to be re-analyzed accordingly.

[2] A boundary vertex is shared by two boundary edges each of which is common to one face only. Interior vertices are shared by edges common to two faces.

One restriction of the approach in (Nasri, 1997a, 1997b) was that the curves interpolated are isolated and must not intersect, hence a challenging problem of generating a surface interpolating a mesh of intersecting curves remains unsolved.

In this paper, we propose a method that can solve this problem by using polygonal complexes. Such a complex is a sequence of panels where each two adjacent ones share one edge only. We studied the recursive subdivision of these complexes and identified conditions for controlling their curves. The mathematical theory involved is established and a number of conceivable applications are outlined. This includes

(1) the interpolation of arbitrary or rectangular meshes of curves by a subdivision surface,
(2) the insertion of edges along which the limit surface can be trimmed, split or joined with different level of continuity, and
(3) and the generation of free-form curves by polygonal complexes.

Based on the results obtained, the algorithms in (Nasri, 1997a, 1997b) can be extended to interpolate arbitrary or rectangular networks of predefined curves where the regions inside the interpolated curves do not have to be 4-sided—a requirement by tenor product based CAD systems.

The paper is structured as follows. Section 2 describes the main problems in subdividing a polygonal complex. Section 3 describes the types of panels used in the construction of symmetric polygonal complexes whose convergence to curves are discussed in Section 4. Section 5 outlines some applications of the theory involved where two approaches can be used to generate subdivision surfaces through a mesh of predefined curves. Finally, in Section 6 we draw conclusions and further work.

2. Subdivision of polygonal complexes

Let us start by defining what is a polygonal complex.

Definition 1. A polygonal complex is defined as a sequence of polygons or panels $(q_i)_{a\leqslant i\leqslant n}$ with the property that every two panels q_j and q_{j+1} have one edge in common. If the two panels q_1 and q_n share an edge the complex is annular, otherwise it is a strip complex.

Fig. 8 shows an example of such a complex.

Recursive subdivision of a polygonal complex consists of applying the subdivision rules to the panels and the shared edges of the complex but not to the vertices. Consequently, there are no V-faces generated in the subdivision of a polygonal complex.

The question to be answered is *what is the limit of subdivision of a polygonal complex*?

Let us examine first the case where all panels of the complex are 4-sided. One may view such a complex as a part of a mesh of a B-spline tensor product surface. Clearly this mesh converges to its corresponding tensor product B-spine surface where the complex converges to one parameter line. As for a general polygonal complex, the limit curve is not in general G^1 at the centroid of the n-sided faces but simply C^0; the curve is a collection of quadratic B-spline pieces with more and more pieces generated after each

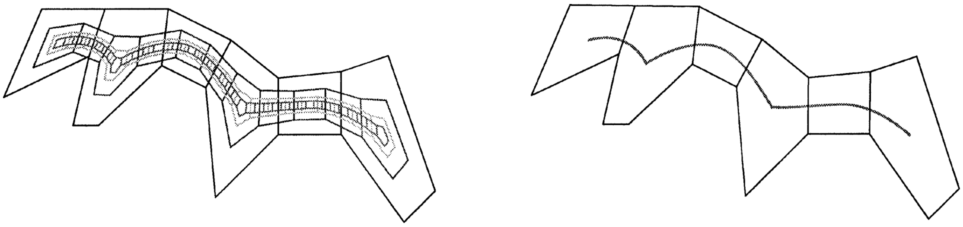

Fig. 3. Convergence of non symmetric polygonal complexes. Three successive subdivisions (left) and the limit curve (right) are shown. Note that G^1 is not achieved at the n-sided non symmetric face.

level of subdivision that joins with C^0 continuity at the centroid of n-sided ($n \neq 4$) panels. The reason is that, with more divisions, we observe that

(1) more and more 4-sided panels are generated and
(2) the number of the reflected faces is invariant.

As a result, the quadrilateral case can then be applied almost everywhere. Fig. 3 gives a counter example showing that G^1 is not achieved in general. [3]

Since the limit curve of a complex is defined regressively, its control polygon is not known in general. However, if the panels of the complex enjoys some symmetry, the control polygons can be defined and hence its limit curve is predictable.

3. Symmetric polygonal complexes

In this section we identify types of panels that can be used in constructing symmetric polygonal complexes whose having predictable limit curves can be predictable. We define two types of panels that can be used in a polygonal complex: single and double reflected panels.

Definition 2. Given a segment $[AB]$. The set

$$Cb^k(A, B) = \{c_i\colon\ i = 1, 2, \ldots, k\}$$

of Chebychev points on $[AD]$ is given by:

$$c_i = \frac{(1+\beta_i)A + (1+\beta_i)B}{2}, \tag{3.1}$$

where the values β_i are given by

$$\beta_i = \frac{\cos\frac{(2i-1)\pi}{2k}}{\cos\frac{\pi}{2k}}. \tag{3.2}$$

Observe that $A = c_1$ and $B = c_k$; they are called Chaikin end-points.

Let n be even, and let f be an n-sided face. Write $n = 2m$. The set of vertices $(v_i)_{1\leqslant i\leqslant m}$ of f are called *original* vertices and the rest are *reflected* vertices. Furthermore, two

[3] An analytic proof for this is possible but not too instructive (Reif, 1996).

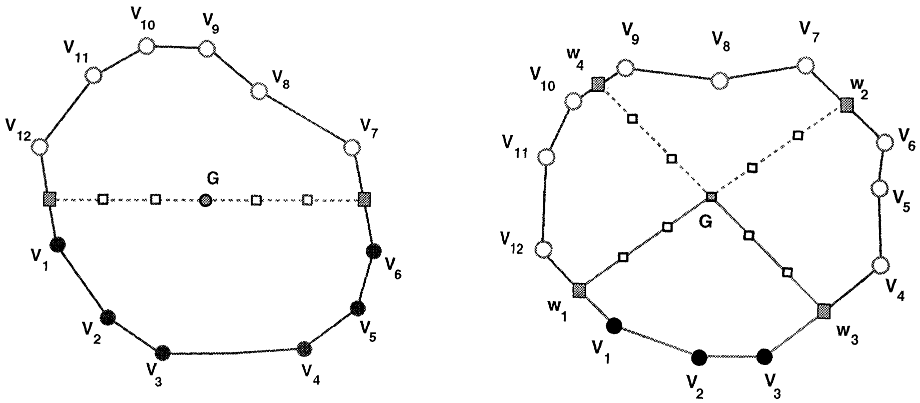

Fig. 4. Two main types of panels in a symmetric polygonal complex: Single panel (left) and double-reflected (right). Original vertices are shown in solid circles, reflected in hollow circles, Chebychev points in hollow squares, and the centroids in shaded squares.

vertices v_i and v_j of an n-sided, n is even, face are called *opposite* vertices, if $i + j = n + 1$. See Fig. 4.

Definition 3. Let n be even, $n = 2m$. An n-sided face f is called single-reflected about a segment $\widehat{e}$ if:

(1) The vertices of $\widehat{e}$ are the midpoints w_1 and w_m of the edges $v_1 v_n$ and $v_m v_{m+1}$, respectively.

(2) Every two opposite vertices, (v_i, v_{n+1-i}) are symmetric about the corresponding Chebychev point $c_i \in Cb^m(w_1, w_m)$.

The segment $\widehat{e}$ is called a *mid-segment* of f and joins the two Chebychev end-points c_1 and c_k. The edges $v_m v_{m+1}$ and $v_{2m} v_1$ are called *contact* edges. [4]

Consider an n-sided face f of vertices $(v_i)_{1 \leqslant i \leqslant n}$, where $n = 4m$, and let G be its centroid. Let $(w_i)_{1 \leqslant i \leqslant 4}$ be the midpoints of the edges $v_{4m} v_1$, $v_{2m} v_{2m+1}$, $v_m v_{m+1}$ and $v_{3m} v_{3m+1}$, respectively. These edges will be referred to as *contact* edges. Furthermore, two contact edges are called *opposite* if their midpoints are collinear with the centroid G, otherwise they called adjacent.

Definition 4. An n-sided, $n = 4m$, face f is called double-reflected about the two segments $w_1 w_3$ and $w_2 w_4$ defined as above if f is single-reflected about $w_1 w_3$ and $w_2 w_4$.

The two segments $w_1 w_2$ and $w_3 w_4$ are called the mid-segments of f and they intersect at G. The vertices of the sets $(v_i)_{1 \leqslant i \leqslant m}$ form the set of original vertices of f and the rest are called reflected vertices. Fig. 4 shows an example of such a face where the contact edges $v_1 v_{12}$, $v_6 v_7$ are opposite edges and $v_1 v_{12}$, $v_3 v_4$ are adjacent.

[4] These will be edges that can be shared with other panels as explained later.

It should be noted that a single-reflected face is a special case of a double-reflected case where both of its mid-segments are collinear but each face plays a different role in terms of the limit curve of a complex.

Definition 5. A symmetric polygonal complex is a polygonal complex whose panels are either single- or double-reflected and whose shared edges are only contact edges.

In the sequel we drop the term symmetric and assume (unless otherwise stated), that all polygonal complexes are symmetric. The *mid-polygon* of such a complex is the piecewise polygon whose vertices are the mid-points of the shared edges. If the two shared edges of

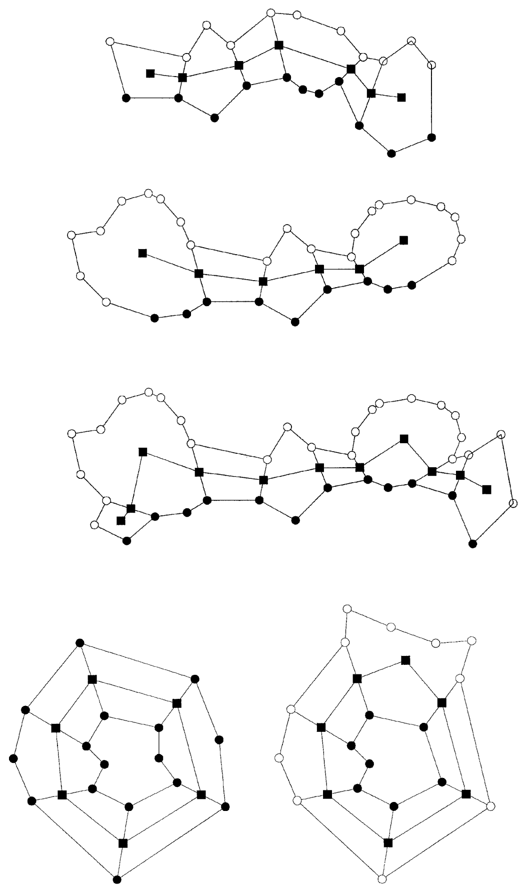

Fig. 5. Various symmetric polygonal complexes and their mid-polygons: Starting from top: a strip whose panels are all single-reflected, a strip with double-reflected end-panels, a general form of a strip complex, an annular complex with single-reflected panels (left), and an annular complex with one double-reflected (right) are shown with original vertices (solid circles), reflected vertices (hollow circles), and mid-polygons vertices (solid squares).

a double-reflected panels are not opposite than its centroid is a common end-point of two pieces of the mid-polygon. In the case of a strip complex, the centroids of the end-panels are the end-points of the mid-polygon.

Fig. 5 give examples of various types of symmetric complexes and their corresponding mid-polygons.

4. Convergence of symmetric polygonal complexes

We begin by showing that a single or a double-reflected panel remains invariant under Doo–Sabin subdivision.

Lemma 1. *Let f_i be a single-reflected face and e_i be its mid-face segment. The F-face of f_i is also single-reflected and its mid-face segment is the E-edge of e_i.*

Proof. We need to show that the midpoints $\widehat{c}_1$ and $\widehat{c}_m$ of the subdivided contact edges are Chaikin end-points of the edge e_i and the subdivided vertices are symmetric about the Chebychev points of the E-edge of e_i as depicted in Fig. 6. See Appendix A for the proof. □

Since a double-reflected panel can be thought of as single-reflected about its two mid-segment, one can easily conclude the following:

Lemma 2. *If a face f_i is a double-reflected about its mid-face segments w_1w_3 and w_2w_4, then its F-face is also double-reflected about the E-edges of w_1w_3 and w_2w_4, respectively.*

Proof. f_i is a single-reflected about each of its mid-face segments. Using Lemma l, its F-face will also be single-reflected about the E-edges of these segments, hence it is double-reflected about them. □

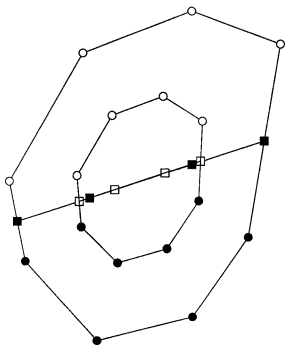

Fig. 6. Invariance of a single reflected panel under subdivision. The Chebychev points c_i of the original (solid squares) and those $\widehat{c}_i$ of the subdivided face (hollow squares) are shown.

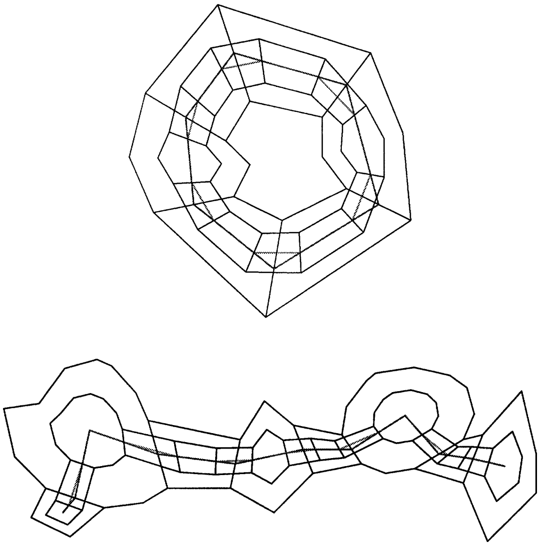

Fig. 7. Invariance of a symmetric polygonal complex under Doo–Sabin: Annular case (top), strip case (bottom).

Applying these lemmas to a symmetric complex yields the following:

Lemma 3. *Let S be a symmetric polygonal complex and M be its mid-polygon. Let $\widehat{S}$ be one step of Doo–Sabin subdivision of S. Then $\widehat{S}$ is a symmetric polygonal complex and its mid-polygon $\widehat{M}$ is the Chaikin subdivision of M.*

Proof. Let us take the case where S is annular. Each face $\hat{f}_i$ of $\widehat{S}$ is either (1) an F-face of a face f_i of S, or (2) an E-face of an edge e_i of S (see Fig. 7). In the first case, $\hat{f}_i$ is symmetric and its mid-segments is the E-edge of that of f_i—a direct result of Lemma 1. Since the E-faces are symmetric then $\widehat{S}$ is also symmetric. The mid-segment of an E-face will join the mid-points of its two shared edges, which are Chaikin endpoints of e_i, therefore $\widehat{M}$ is the Chaikin subdivision of M. The strip case can be handled in an analogue manner. □

The above results leads to the following essential theorem.

Theorem 1. *Let Q_0 be a strip complex, and M_0 be its mid-polygon. Denote by Q_k the kth subdivision of Q_0 then*

$$\lim_{k\to\infty} Q_k = c,$$

where c is the piecewise quadratic B-spline curve of M_0 which interpolates the centroid of each panel of Q_0 with C^1 continuity if its two contact edges are opposite and with C^0 only if they are adjacent.

Proof. Let M_i be the ith Chaikin subdivision of M_0. We have

$$\lim_{i\to\infty} M_i = c,$$

where c is the corresponding quadratic B-spline curve. To begin with, consider first a single complex where all panels are single reflected. This means that there is a 1–l correspondence between the points of the mid-polygon of S_i and the points of each of the outer polygons P_i^r and p_i^o whose sets of vertices are, respectively, V_o and V_r—the sets of the original and reflected vertices. Accordingly, for a parameter value u we can define $S_j^o(u)$ and $S_j^r(u)$ to be the corresponding points on p_i^o and p_i^r, respectively. We have to prove that for any $\varepsilon > 0$, there exists j such that $\|S_j^o(u) - c(u)\| < \varepsilon$ and $\|S_j^r(u) - c(u)\| < \varepsilon$ for all u.

Denote by d_i^j the diameter of a panel q_i^j of Q_i and by

$$\|d_i\| = \max_j\bigl(d_i^j\bigr). \tag{4.1}$$

Since the second largest eigenvalues of the subdivision matrix $\lambda = 1/2$ (Doo and Sabin, 1978), we have:

$$\|d_{i+1}\| = \frac{\|d_i\|}{2} = \frac{d_0}{2^i}. \tag{4.2}$$

Given an $\varepsilon > 0$, the convergence of M_i ensures that there exists a j_0 such that

$$\bigl\|M_j(u) - c(u)\bigr\| < \varepsilon$$

for all $j > j_0$ and all u. Using Lemma 1, for each M_j, there exists a subdivided complex S_j whose mid-polygon is M_j. In addition, $M_j(u)$ must belong to a leg l_m of M_j, which is also a mid-segment of a certain panel q_m of S_j. On this panel,

$$\bigl\|S_r^o(u) - c(u)\bigr\| < \bigl\|S_r^o(u) - M_j(u)\bigr\| + \bigl\|M_j(u) - c(u)\bigr\|,$$

which gives:

$$\bigl\|S_r^o(u) - c(u)\bigr\| < \frac{d_0}{2^j} + \varepsilon.$$

One may then choose j such that

$$\frac{d_0}{2^j} < \varepsilon$$

and hence

$$\bigl\|S_r^o(u) - c(u)\bigr\| < 2\varepsilon.$$

The other inequality can be similarly deduced which completes the proof in the case of a strip with single-reflected panels only. As for the general case where a strip may include double-reflected panels, two cases are considered depending on whether the contact edges of this panel are opposite or adjacent. In the case of opposite contact edges, the 1–1 correspondence is maintained and the centroid is interpolated with C^1 continuity since the panel is single-reflected about the corresponding mid-segment. Otherwise, the limit curve will pass through the centroid and will be tangential to the two mid-segment e_1 and e_2. Since e_1 and e_2 are not collinear, C^1 is not achieved and the curve is C^0 only. □

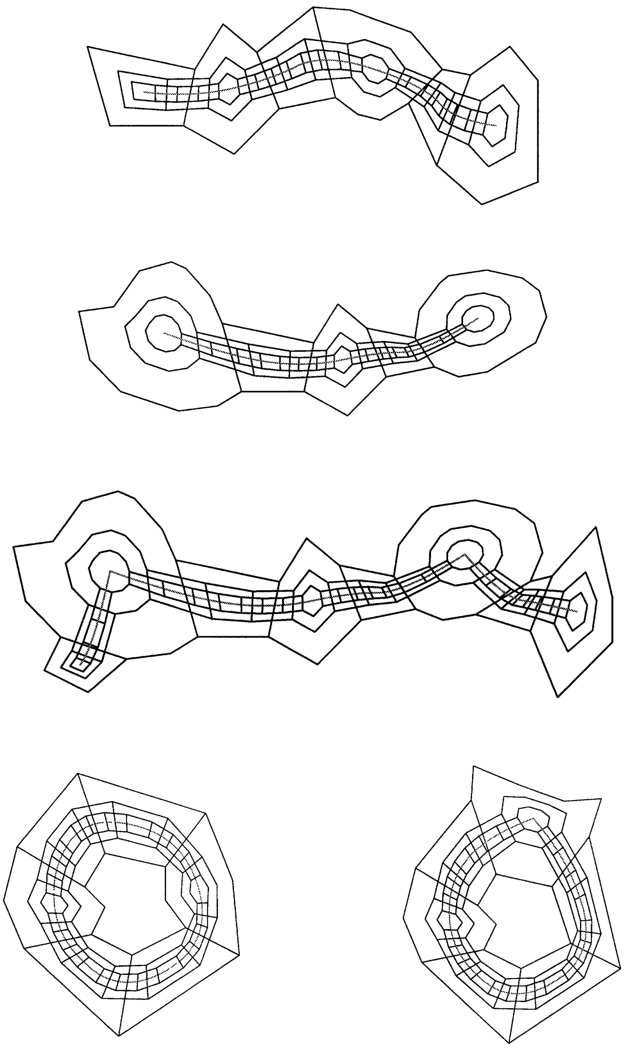

Fig. 8. Convergence of polygonal complexes. Each complex is shown with two successive subdivision and its limit curve. The top three complexes illustrate various strip cases and the bottom two depict annular cases.

Fig. 8 shows the limit curves of various shapes of polygonal complexes. Based on this theorem the following corollary can be devised:

Corollary 1. *Let* $(Q_i)_{1\leqslant i\leqslant m}$ *be* m $(m \leqslant 4)$ *symmetric complexes sharing one double-reflected panel* p_0*. Then, the complexes converge to* m *corresponding curves* $(c_i)_{\leqslant i\leqslant m}$ *meeting at the centroid of* p_0 *such that every couple* (c_i, c_j) *meets with* C^1 *continuity if the* *contact edges of Q_i and Q_j are opposite, and with C^0 continuity only if the contact edges are adjacent.*

Proof. Consider the case of $m = 4$ first. Let g_i, w_1w_3 and w_2w_4 be the centroid, and the two mid-segments of p_0, respectively. Denote by Q_{ij} the piecewise strip made of Q_i and Q_j and by c_{ij} its piecewise curve which is made of c_i, and c_j. Clearly, p_0, being single reflected about w_1w_3 and w_2w_4, can be considered as a single panel of the strip Q_{13} and Q_{24}. These strips will converge to two curves c_{13} and c_{24}, respectively, and both of them pass through g_i with C^1 continuity. On the other hand, since w_1w_3 and w_2w_4 are not

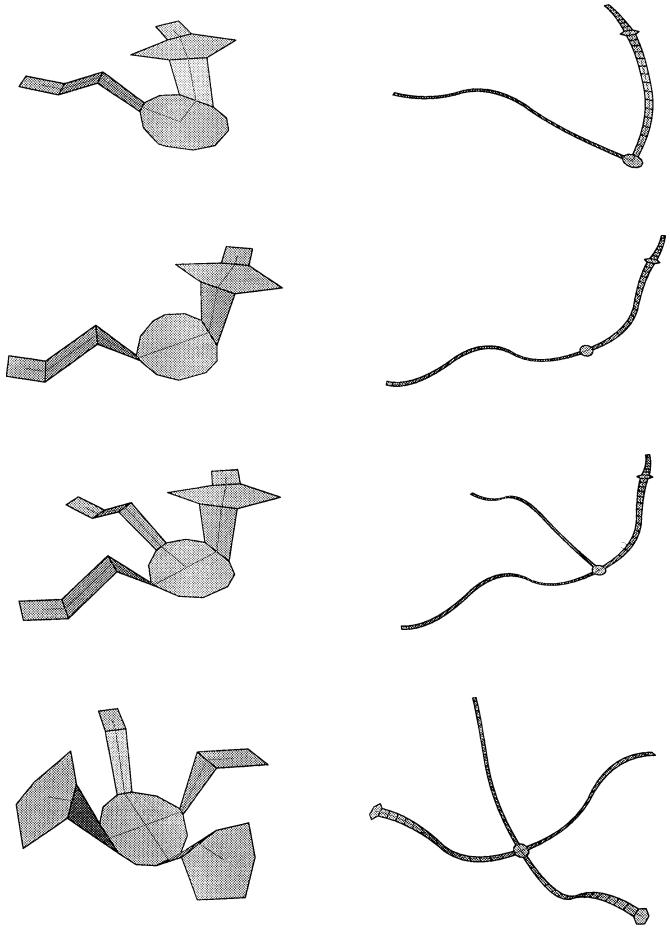

Fig. 9. Convergence of polygonal complexes sharing one double-reflected panel. From top: two complexes giving two limit curves with C^0 joint, two complexes giving two limit curves with C^1 joint, three complexes giving three limit curves, and four complexes giving four limit curves. In all figures, complexes and mid-polygons arc shown on left, their third subdivisions and limit curves are shown on right.

collinear, the strips Q_{23} and Q_{14} converge to two piecewise curves c_{23} and c_{14}. The curve c_{23} is made of c_2 and c_3 whose tangents at g_i are not collinear, thus C^0 continuity is only achieved.

For $m < 4$, one can assume the existence of one or two more complexes(s) sharing p_0 and then apply the 4 complexes case which completes the proof. □

The following situations are then possible:

(1) Two complexes can share a double-reflected panel. If the two contact edges with p_0 are opposite, the two limit curves are C^1 at the centroid of p_0, otherwise they are only C^0 as depicted in Fig. 9.

(2) Three complexes may share a double-reflected panel. Two of the three contact edges must be opposite and hence two of the limit curves meet with C^1 but the third with C^0 only as shown in Fig. 9.

(3) Finally, four complexes can share a double-reflected panel. The complexes sharing this panel with opposite contact edges will have their limit curves meeting with C^1, and those sharing it with adjacent contact edges with C^0 as shown in Fig. 9.

5. Applications

There are a number of conceivable applications of the proposed method. The following sections discuss some of them.

5.1. Curve interpolation

One important application of the proposed method is the generation of a subdivision surface through a set of arbitrary meshes of predefined curves. The problem can be stated as follows:

Given a polyhedral network P and a set of r tagged control polygon $(cp_i)_{1\leqslant i\leqslant r}$, how to force the limit surface generated from P to interpolate the B-spline curves (c_i) of (cp_i).

One major issue to solve such a problem is how to define these curves. Two approaches can be used: the *polygonal* approach and the *Complex approach*.

5.1.1. The polygonal approach

One approach is to start with an initial polyhedral network P_0 and define the curves by some tagged control polygons whose vertices and edges are chosen from those of P_0. The curve interpolation problem can then be solved by using a **polygonal approach** which can be regarded as an extension of the method suggested in (Nasri, 1995, 1997a, 1997b). Basically, the idea consists of constructing polygonal complexes, one for each curve, by modifying some panels of the initial polyhedron or its first subdivision depending on whether the curve to be interpolated is a boundary or an interior one. Fig. 10 shows a polyhedron and a tagged control polygon. The limit surface with and without curve interpolation is shown. Initially the method suggested in (Nasri, 1997a, 1997b) could not handle intersecting curves but the use of polygonal complexes as proposed in this paper makes it achievable. This is to be further investigated.

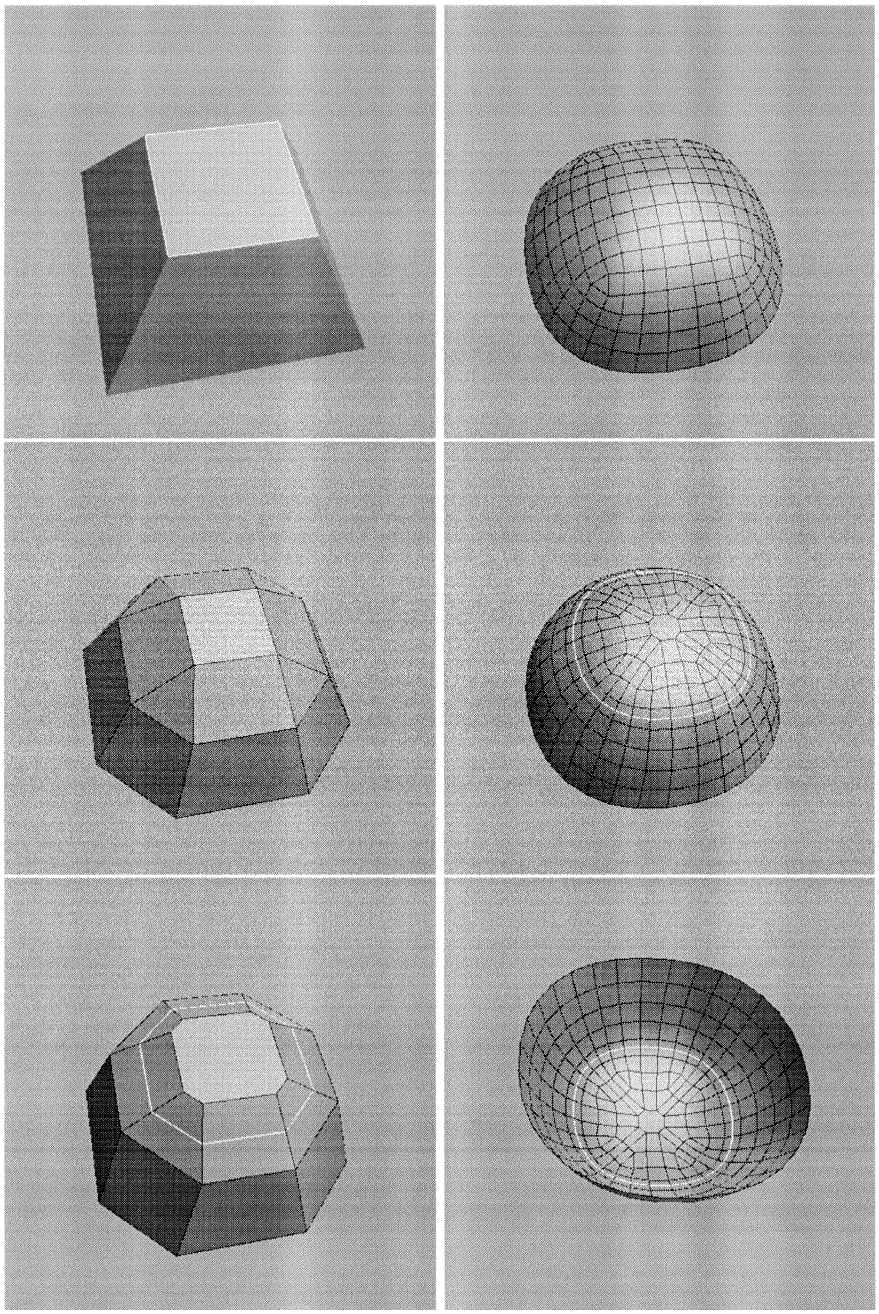

Fig. 10. Left column from top: A configuration with a tagged control polygon and its first subdivision before and after strip construction. Right column from top: the limit surface without curve interpolation, and two views of the surface with curve interpolation.

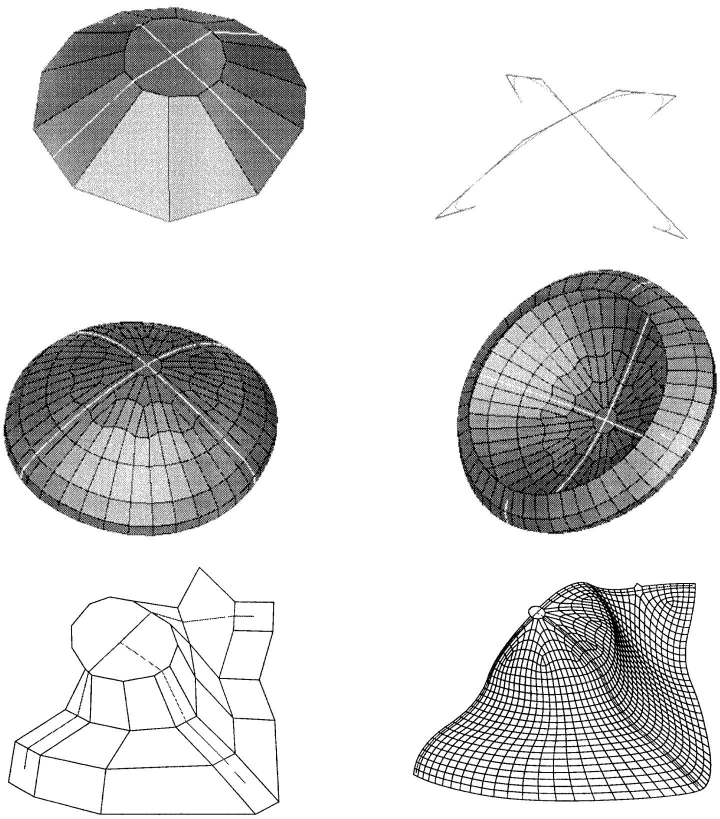

Fig. 11. Interpolating intersecting curves: A surface network containing four strip complexes and their mid-polygons (top left), the limit curves and their polygons (top right), two views of the corresponding limit surface (middle), another surface network containing two strip complexes (one boundary and one interior) and their mid-polygons (bottom left), and the corresponding limit surface (bottom right).

5.1.2. The complex approach

A different approach to interpolating curves by subdivision surfaces consists of using the **complex approach** as follows. First, the interpolated curves should be designed by polygonal complexes whose mid-polygons control the shape of these curves. To do this, the user simply sketches out the control polygons of the curves to be interpolated. Next, a panel p_i, one for each leg l_i of the control polygons, is designed such that the mid-segment of p_i is l_i. For this the user has to choose the position and the number of *original* vertices of p_i. The reflected vertices needed to complete the panel are constructed automatically. Note that the vertices of the panels (whether original or reflected) of the complexes can be moved around with the only constraint that they remain symmetric about their corresponding Chebychev points. However in some cases the polygonal complexes may not alone make a polyhedral network for a limit surface because of the possible existence of 2-valent vertices

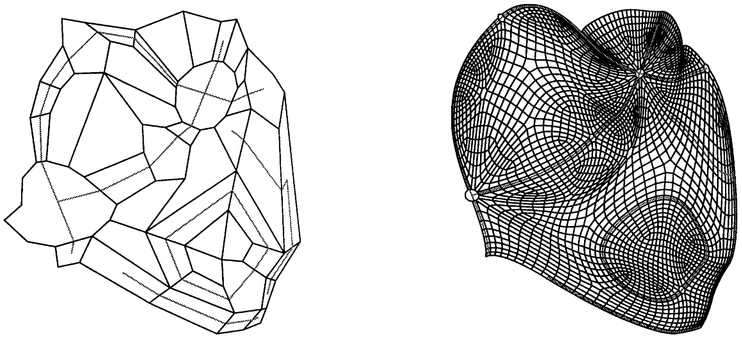

Fig. 12. Interpolating curves which correspond to different types of polygonal complexes: Initial mesh indicating strips and their mid-polygons (left), and the corresponding surface with the interpolated curves (right) are shown.

of certain double-reflected panels. Such vertices are not allowed in the definition of a polyhedron where all vertices must be n-valent with $n > 2$. To modify their valences, the user will have to complete the polyhedron by interactively filling in some vertices inside the regions enclosed by the complexes. This is a design process which is basically similar to designing a polyhedral network for a subdivision surface. Essentially, the vertices needed inside the regions can be sparse and just enough to increase the valence of the 2-valent vertices. One solution that could be used as a default construction in an interactive design system consists of the following:

(1) Compute an *average* vertex from the centroids of all panels bounding a region between interpolated curves.
(2) Insert the average vertex in the set of vertices of the constructed polyhedron.
(3) Connect all 2-valent vertices bounding the region to the average vertex.
(4) Insert their corresponding faces in the topology of the polyhedron defining the limit surface.

Note that additional vertices can be inserted in the regions but there must be reasons for doing so such as controlling the shape of the surface inside those regions and it is eventually a designer decision.

Having done that, theorem 1 guarantees that the limit surface of the constructed polyhedron will interpolate the limit curves of the complexes. Thus, if the network contains initially, two complexes sharing a double-reflected panel, two curves meeting at an interior point can be interpolated by the limit surface. With three complexes, an interior curve can intersect a boundary one where the contact point is naturally C^0 only. With four complexes, four intersecting curves can be interpolated. Figs. 11 and 12 provide examples of interpolating intersecting curves using this approach.

5.1.3. Interpolating surfaces

Using the approaches suggested above, recursive subdivision surfaces through rectangular or arbitrary networks of curves where n $(n < 4)$ curves may meet can be generated.

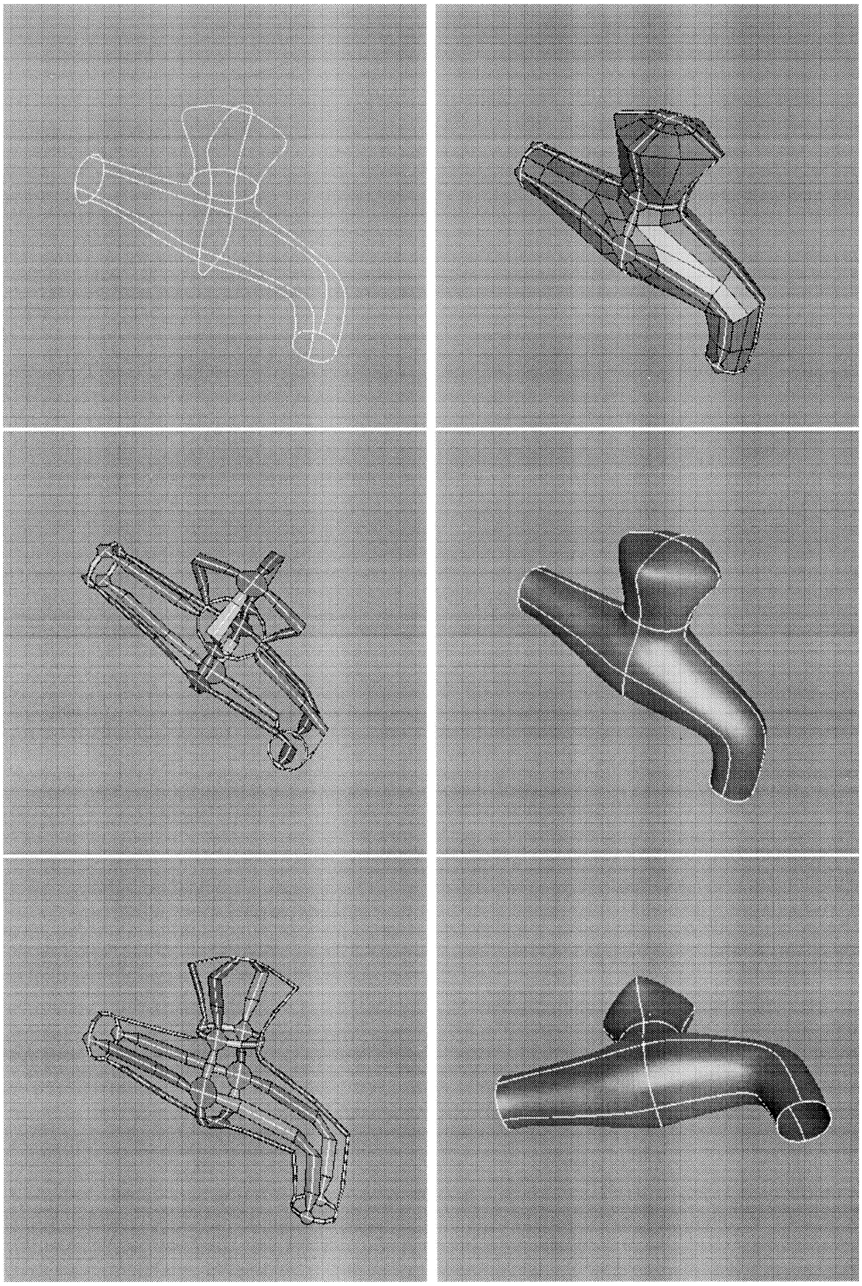

Fig. 13. A tap surface interpolating an arbitrary mesh of curves with various n-sided ($n = 3, 4, 5$) regions. Left column from top: A mesh of given curves and two views of the polygonal complexes defining these curves. Right column from top: A network incorporating these complexes and two views of the corresponding limit surface interpolating the given curves.

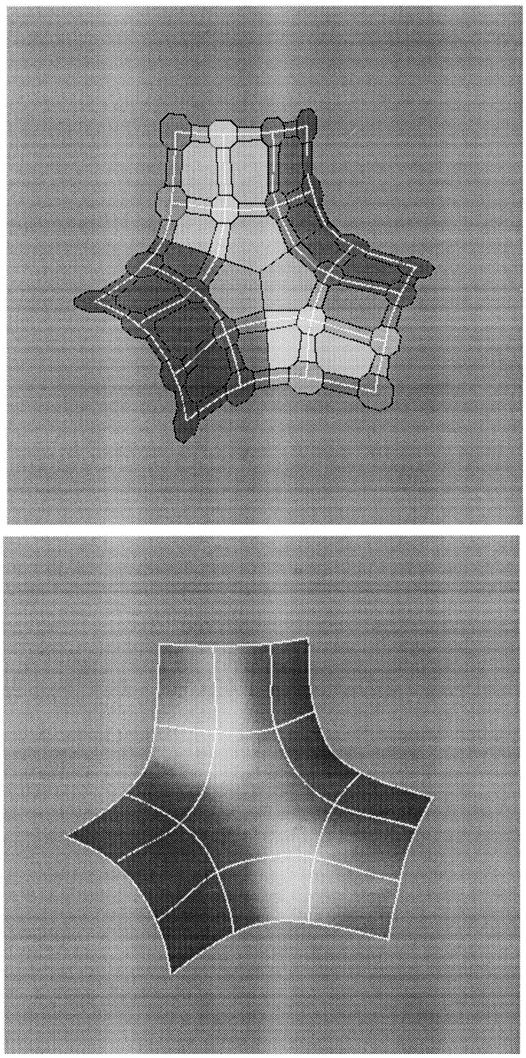

Fig. 14. A surface interpolating a mesh of curve with a 6-sided region enclosed by these curves: A polyhedron with polygonal complexes (top) and limit surface interpolating the predefined curves (bottom).

Fig. 12 shows a network with various polygonal complexes (top right) and its limit surface passing through the corresponding network of arbitrary curves.

Compared to the tensor-product based CAD systems, the scheme has the advantage of incorporating n-sided regions between interpolated curves as depicted in Fig. 13 where 3-, 4-, 5-sided regions are shown. Fig. 14 shows also an example of a 6-sided region between interpolated curves.

As for the quality of the resulting interpolating surfaces, they are smooth surfaces, being limits of a subdivision process. They lie within the convex hulls of their modified or initial networks.

One advantage of the *complex* approach which is used to produce the above figures is that the user has control on the shape of the polygonal complexes defining the interpolated curves. This is possible since they are originally included in the defining network. The shape of the surface across the interpolated curves can be controlled by interactively modifying the position of the vertices of the panels around the their corresponding Chebychev points. Furthermore, the artifact shown in the surfaces of Figs. 11 and 12 along the boundary can be eliminated by a process of boundary modification as suggested in (Nasri, 1987). This was not considered here to clearly show how the complexes are converging to their corresponding curves.

5.2. Edge insertion and trimming

The limit curve of a polygonal complex can be regarded as an edge inserted on the surface as used in (Habib, 1996). Such an edge is a feature line or a shape handle along which a surface can be trimmed or split as indicated in Fig. 15. Continuity across an inserted edge can be controlled. For example two polyhedra can be joined along a specific edge with C^1 if they both share the same polygonal complex. Twisting the panels of these complexes along their mid-segments can reduce the joint to C^0.

5.3. Curve generation

One further application that is worth pursuing is the generation of free-form curves by polygonal complexes. This has several advantages such as incorporating curves in the network of a given surface. The panels of the corresponding complex can be used to define tangent planes along this curve. Further research is still needed to establish similar results to the control-polygon case such as parameterization of a limit curve, degree elevation, curvature, and other geometric properties.

6. Conclusions and further work

In this paper, we have presented a method for subdividing polygonal complexes and identified conditions on their panels to control their limit curves—a property not possible for a general complex. The general theory established has a number of applications in CAGD amongst which are the curve interpolation by the limit surface, edge insertion and trimming, and free-form curve generation.

In curve interpolation, the scheme is capable of generating surfaces through predefined rectangular meshes or arbitrary networks of curves. This feature makes the whole subdivision scheme more attractive and more practical in surface modeling and computer graphics. Furthermore, the scheme has the advantage of generating n-sided, where n is not necessarily 4, regions enclosed by the limit curves—a limitation of most tensor-product based CAD systems. Each interpolated curve can play the role of a feature line on the limit

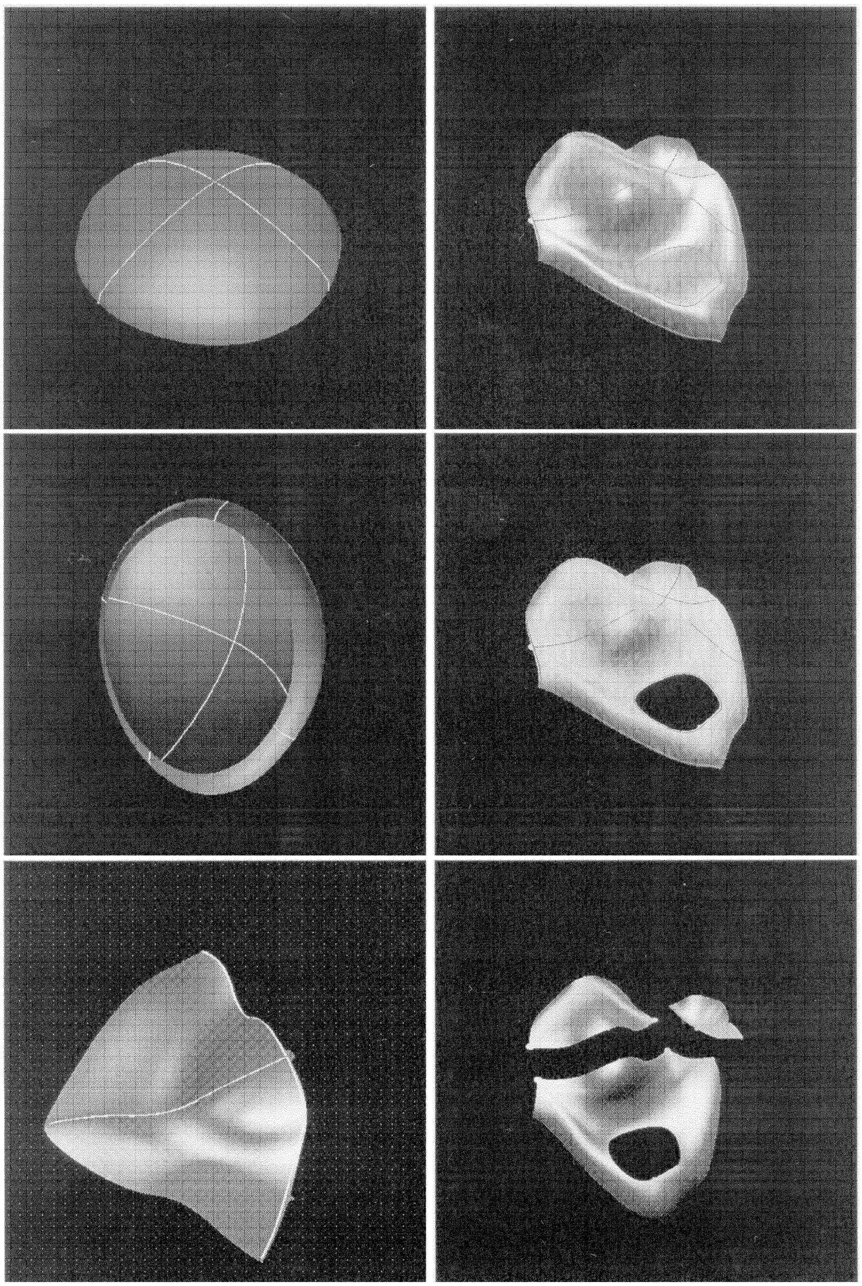

Fig. 15. Left: Color shaded pictures of the surfaces in Fig. 12 interpolating intersecting curves on the interior (top and middle) and on the boundary Fig. 11. Right: Trimming of a subdivision surface along interpolated curves. Surface of Fig. 12 (top) trimmed along the interior closed curve (middle), and then split into 3 pieces (bottom).

surface as an inserted edge along which the surface can be split, trimmed or joined with another with various level of continuity.

One limitation of the proposed scheme is that no more than four curves are allowed to intersect at an interior point of the surface. Such a limitation is the subject of a subsequent paper. Designing shape handles that control the shape of the surface across an interpolated curve and the generation of free-form curves by polygonal complexes are also subjects for further work in this direction. Finally, although Doo–Sabin scheme was mainly used in the proposed method, the extension to higher order subdivision surfaces can be inspired. This is also currently under investigation.

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Appendix A

Proof of Lemma 1. The vertices (w_i) of the F-face of f_i are given by:

$$\begin{bmatrix} w_i \\ \cdot \\ w_i \\ \cdot \\ w_m \\ w_{m+1} \\ \cdot \\ w_{n-2i+1} \\ \cdot \\ w_n \end{bmatrix} = M \times \begin{bmatrix} v_i \\ \cdot \\ v_i \\ \cdot \\ v_m \\ v_{m+1} \\ \cdot \\ v_{n-2i+1} \\ \cdot \\ v_n \end{bmatrix},$$

where M is given by:

$$M = \begin{bmatrix} \alpha_0 & \cdot & \alpha_{i-1} & \cdot & \alpha_m & \alpha_{m+1} & \cdot & \alpha_{m-1} & \cdot & \alpha_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{i-1} & \cdot & \alpha_{n-2i+1} & \cdot & \alpha_{m-1} & \alpha_{m-i+1} & \cdot & \alpha_0 & \cdot & \alpha_i \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{m-1} & \cdot & \alpha_{m-i} & \cdot & \alpha_0 & \alpha_1 & \cdot & \alpha_{m-i+1} & \cdot & \alpha_m \\ \alpha_m & \cdot & \alpha_{m-i+1} & \cdot & \alpha_1 & \alpha_0 & \cdot & \alpha_{m-i} & \cdot & \alpha_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha i & \cdot & \alpha_0 & \cdot & \alpha_{m-i+1} & \alpha_{m-i} & \cdot & \alpha_{n-2i+1} & \cdot & \alpha_{i-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_1 & \cdot & \alpha_i & \cdot & \alpha_{m-1} & \alpha_m & \cdot & \alpha_{i+1} & \cdot & \alpha_0 \end{bmatrix}.$$

Here we are assuming that $m = n/2$ and $i \geqslant m/2$. Other cases can be easily inspired. Let c_1 and c_m be the midpoints of $v_1 v_n$ and $v_m v_{m+1}$, and $\widehat{c}_1, \widehat{c}_m$ be the midpoints of $w_1 w_n$ and $w_m w_{m-1}$, respectively. We have to prove two things:

(1) that E-edge of the mid-segment $c_1 c_m$ is $\widehat{c}_1\widehat{c}_m$ and,

(2) that the (w_i) are symmetric with respect to the Chebychev points $(\widehat{c}_i)$ defined on $\widehat{c}_1\widehat{c}_m$ (see Fig. 6).

We have to prove that w_i and w_{n-2i+1} are symmetric about $\widehat{c}_i$ which is given by

$$\widehat{c}_i = \sum_{k=1}^{m-i} c_k(\alpha_{i-k} + \alpha_{i+k-1}) + \sum_{k=m-i-1}^{i-1} c_k(\alpha_{i-k} + \alpha_{n-i-k+1}) + \sum_{k=1}^{m} c_k(\alpha_{k-1} + \alpha_{n-i-k+1}).$$

Using the following identities

$$\alpha_i + \alpha_j = \frac{1}{8m}\left(6 + 4\cos\frac{\pi(i-j)}{2m}\cos\frac{\pi(i+j)}{2m}\right), \tag{A.1}$$

$$\alpha_i + \alpha_0 = \frac{1}{8m}\left(6 + 4\cos^2\frac{\pi(i-j)}{2m} + \frac{1}{4}\right), \tag{A.2}$$

we get

$$\begin{aligned}
\alpha_{i-k} + \alpha_{i+k-1} &= \frac{1}{8m}\big[6 + 4\cos(2k-1)x\cos(2i-1)x\big],\\
\alpha_{i-k} + \alpha_{n-i-k+1} &= \frac{1}{8m}\left[6 + 4\cos(n-2k-1)x\cos(n-2i-1)x + \frac{1}{4}\right],\\
\alpha_{k-i} + \alpha_{n-i-k+1} &= \frac{1}{8m}\big[6 + 4\cos(n-2i-1)x\cos(n-2k-1)x\big].
\end{aligned} \tag{A.3}$$

With $x = \pi/2m$ and $\cos(n-2k-1)x = \cos(2k-1)$, $\widehat{c}_i$ can be written as

$$\widehat{c}_i = \frac{c_i}{4} + \sum_{k=1}^{m}\left[\frac{1}{8m}\big(6 + 4\cos(2k-1)\cos(2i-1)x\big)\right]c_k \tag{A.4}$$

$$= \frac{c_i}{4} + \frac{3}{4m}\sum_{k=1}^{m} c_k + \frac{\cos(2i-1)}{2m}\sum_{2m}^{m} c_k\cos(2k-1)x. \tag{A.5}$$

Using the following equations

$$\sum_{k=1}^{m}\beta_k = 0, \tag{A.6}$$

$$c_k + c_{m-k+1} = \frac{c_1 + c_m}{2}, \tag{A.7}$$

we have

$$\sum_{k=1}^{m} c_k = \frac{m}{2}(c_1 + c_m). \tag{A.8}$$

The other summation can be computed as follows. Using the following identities

$$\cos\bigl(2(m-k+1)-1\bigr)x = -\cos(2k-1)x c_k - c_{m-k+1} = (c_1 - c_m)\beta_k,$$

we compute

$$2\sum_{k=1}^{m} c_k \cos(2k-1)$$

as

$$\sum_{k=1}^{m}\bigl[c_k\cos(2k-1)x + c_{m-k+1}\cos\bigl(2(m-k+1)-1\bigr)x\bigr] \tag{A.9}$$

$$= \sum_{k=1}^{m}(c_k - c_{m-k+1})\cos(2k-1)x \tag{A.10}$$

$$= (c_1 - c_m)\sum_{k=1}^{m}\beta_k\cos(2k-1)x. \tag{A.11}$$

Replacing β_k by its value, the above equation is given by

$$\frac{(c_1 - c_m)}{\cos x}\sum_{k=1}^{m}\cos^2(2k-1)x. \tag{A.12}$$

Putting $\cos^2(2k-1)x$ as

$$\cos^2(2k-1)x = \frac{1}{2}\bigl(1 + \cos 2(2k-1)x\bigr) \tag{A.13}$$

we get

$$\sum_{k=1}^{m}\cos^2(2k-1)x = \frac{1}{m} + \frac{1}{2}\sum_{k=1}^{m}\cos 2(2k-1)x. \tag{A.14}$$

But as

$$\cos 2(2k-1)x = \cos\frac{(2k-1)\pi}{m} = \cos\frac{2k\pi}{m}\cos\frac{\pi}{m} + \sin\frac{2k\pi}{m}\sin\frac{\pi}{m},$$

we have

$$\sum_{k=1}^{m}\cos 2(2k-1)x = \cos\frac{\pi}{m}\sum_{k=1}^{m}\cos\frac{2k\pi}{m} + \sin\frac{\pi}{m}\sum_{k=1}^{m}\sin\frac{2k\pi}{m}. \tag{A.15}$$

Using the two identities

$$\sum_{j=1}^{m}\cos\alpha k = \frac{\sin\frac{m\alpha}{s}}{\sin\frac{\alpha}{2}}\sin\frac{(m+1)\alpha}{2}, \tag{A.16}$$

$$\sum_{j=1}^{m}\sin\alpha k = \frac{\sin\frac{m\alpha}{s}}{\sin\frac{\alpha}{2}}\cos\frac{(m+1)\alpha}{2}, \tag{A.17}$$

it is easy to show that

$$\sum_{k=1}^{m} \cos \frac{2k\pi}{m} = 0, \tag{A.18}$$

$$\sum_{k=1}^{m} \sin \frac{2k\pi}{m} = 0. \tag{A.19}$$

Therefore

$$\sum_{k=1}^{m} \cos 2(2k-1)x = 0,$$

and hence

$$\sum_{k=1}^{m} \cos^2(2k-1)x = \frac{m}{2}$$

replacing this in above, we get

$$2\sum_{k=1}^{m} c_k \cos(2k-1)x = \frac{m}{2}\frac{c_1 - c_m}{\cos x}. \tag{A.20}$$

Finally,

$$\widehat{c}_i = \frac{c_i}{4} + \frac{3}{8}(c_1 + c_m) + \frac{(c_1 - c_m)\beta_i}{8}. \tag{A.21}$$

Using $\beta_{m+1-i} = -\beta_i$, it is easy to show that this yields

$$\frac{3}{4}c_i + \frac{1}{4}c_{m-i+1} \tag{A.22}$$

and thus the $\widehat{c}_i$ are affine maps of the c_i.

This gives

$$\widehat{c}_1 = \frac{3}{4}c_1 + \frac{1}{4}c_m \widehat{c}_m = \frac{1}{4}c_1 + \frac{3}{4}c_m,$$

and thus $\widehat{c}_1$ and $\widehat{c}_m$ are the Chaikin end-points on c_1c_m and hence $\widehat{c}_1$ $\widehat{c}_m$ is the E-edge of c_1c_m. Therefore the $\widehat{c}_i$ are affine maps of the c_i, which completes the proof. □

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