Vor(P), DT(P), and F-rep – towards Optimal Modeling

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Short podcast on Computational Geometry 2020

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Agenda

- Motivation
- Computational Geometry
  - Methodology, Algorithmic Paradigms
  - Voronoi diagram
  - Delaunay triangulation
- Functional Representation
  - Definitions
  - HyperFun
  - Examples

Graphics & Visual Computing

ACM Computing Curriculum
at http://www.computer.org/education/cc2001/final/gv.htm:

The area encompassed by Graphics and Visual Computing (GV) is divided into four interrelated fields:

- ***Computer graphics.***
- ***Visualization.***
- ***Virtual reality.***
- ***Computer vision.***

Computer Graphics

Computer graphics *is the art and science of communicating information using images that are generated and presented through computation. This requires:*

(a) the design and construction of models that represent information in ways that support the creation and viewing of images,

(b) the design of devices and techniques through which the person may interact with the model or the view,

(c) the creation of techniques for rendering the model, and

(d) the design of ways the images may be preserved. The goal of computer graphics is to engage the person's visual centers alongside other cognitive centers in understanding.

„Computer Graphics...

- … can be formulated as a radiometrically „weighted“ counterpart of computational geometry…
- … rendering is done through the application of a simulation process to quantitative models of light and materials to predict/synthesize appearance“
-
- D. Dobkin & S. Teller, 1999

Computer Graphics...

- … must account geometry
- material properties: reflectance/color, refractive index, opacity, and (for light sources) emmisivity
- radiometry
- output for viewing: explicitly or implicitly psychophysics

- by D. Dobkin & S. Teller, 1999

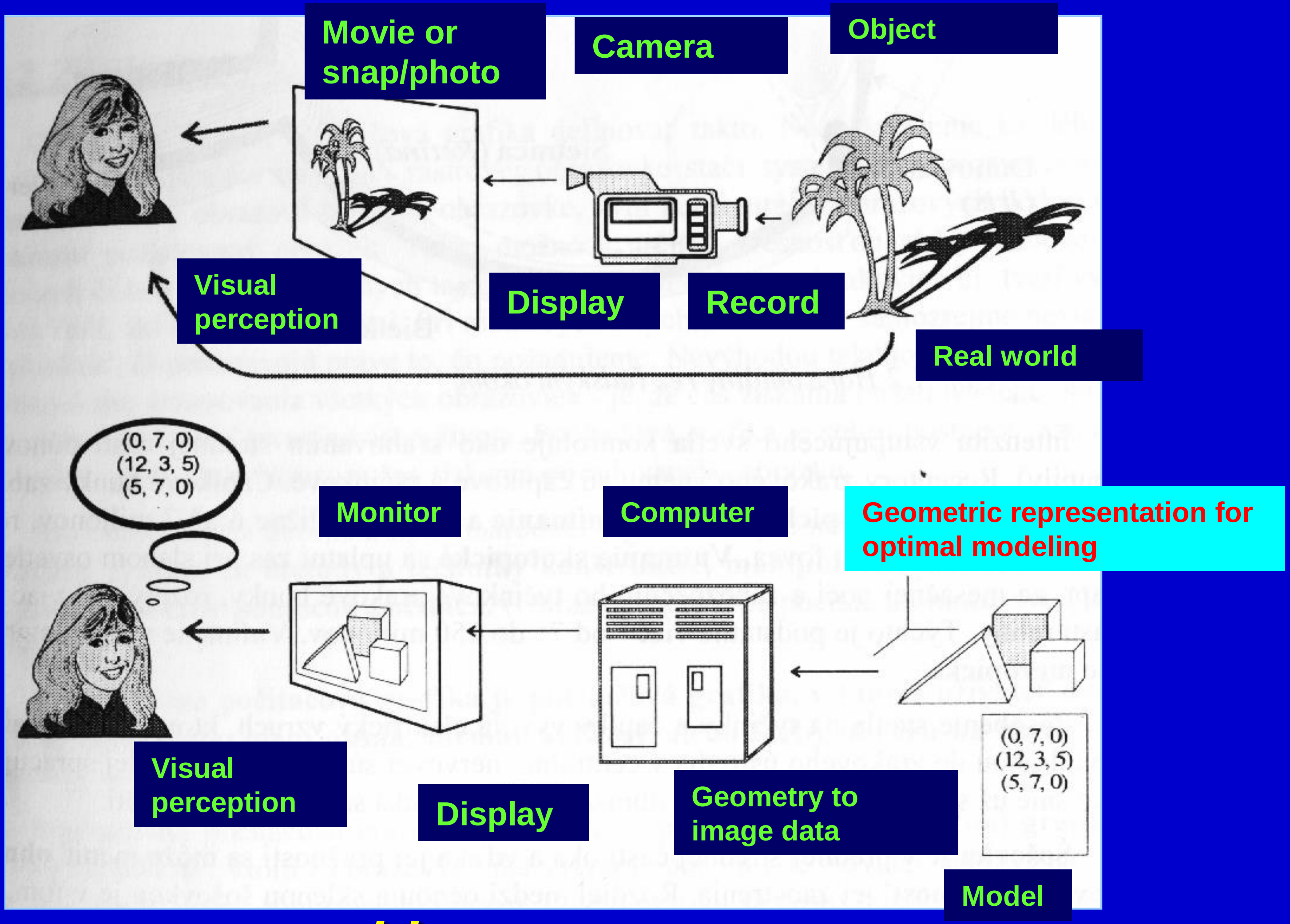
Material Description

Geometric Description

Light Sources Description

Image Synthesis (Rendering Algorithm)

Analogy: photography & computer graphics



- **ISO: Computer graphics:** *methods & techniques for construction, manipulation, storage and displaying pictures using computer.*

Object Representations

- Point-based Graphics
- Curves and Surfaces
- Solid Modeling
  - Boundary Representation (mesh, MR)
  - Spatial Enumeration Models
    - Spatial-Occupancy Enumeration (Voxel)
    - Binary Space Partitioning (BSP) Trees
    - Octrees
  - Constructive Solid Geometry (CSG)
  - Function Representation (F-rep)

Object Representations

- **Elementary Objects**
  - Primitives, regular polyhedra, ...
  - Sweeps
  - Free-form patches
  - (Super-)Quadrics
  - Terrain (DTM, DEM)
  - Fractal Mountains
  - Soft Objects
  - Particle Systems
  - Natural Phenomena…
- **Transformations**
  - linear ones
  - twist, blending ... (Verbiegeoperationen)
  - local operations
- **Combining methods**
  - Boolean Operations with Elementary Objects (CSG)
  - F-rep
  - (Solid Modeler UI)

Math Language Ruptures

- Elementary Arithmetics
- Synthetic Geometry
- Algebra
- <u>Analytic Geometry</u>
- Infinitesimal Calculus
- Iterative Geometry
- Predicate Calculus
- <u>Set Theory</u>

- (based on Kvasz´s epistemologic research, 1996)

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CompGeom Methodology

- *Name coined in PhD thesis by M. I. SHAMOS, 80s*
- *Synthesis and analysis of efficient geometric algorithms, book by SHAMOS-PREPARATA (1985)*
- *Synthesis – algorithmic strategies, paradigms, metaphors, principles...*
- *Analysis – model of computation, problem complexity, brute force algorithm, efficient algorithms, optimal solution (LEDA)*
- *Real RAM model and worst-case complexity today*

CompGeom Assumptions I

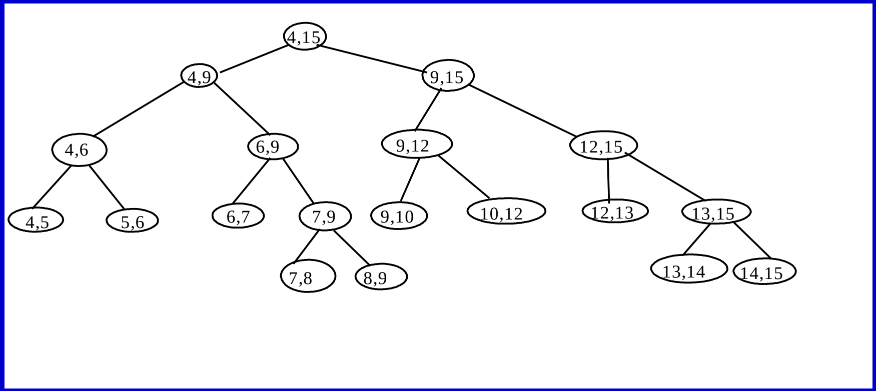
- Euclidean d-dimensional space (d = 2, 3, …), sets S
- Typically, a set is given by linear equation
- a1*x1 + ... + ad*xd = b
- … and other conditions (e.g. polyhedra, mesh, halfplanes)
- Set operations
- 1. MEMBER(u,S)
- 2. INSERT(u,S)
- 3. DELETE (u,S)

CompGeom Assumptions II

- Let {S1, S2, ..., Sk} is a system of pairwise disjoint sets

- 4. FIND(u) in {S1, S2, ..., Sk}
- 5. UNION(Si, Sj; Sk)

- For ordered sets:



- 6. MIN(S)
- 7. SPLIT(u,S) S2 = S - S1
- 8. CONCATENATE(S1, S2)

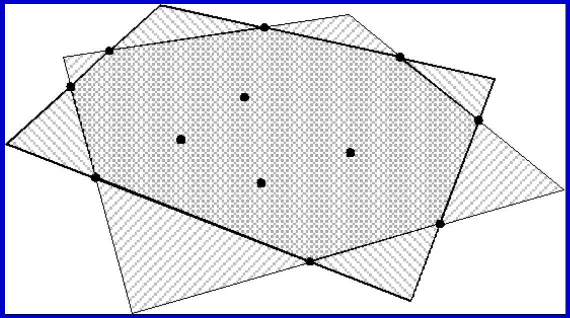
- E.g. VOCABULARY supports MEMBER, INSERT, DELETE
- PRIORITY QUEUE supports MIN, INSERT, DELETE

CompGeom Methodology

- *Real RAM model, unit cost operations, real numbers*
- *Worst-case complexity - the usual (Knuth) notation:*
- *· O(f(N)) means the set of all functions g(N) such that there exist positive constants C and M with*
  - *$| g(N) | < Cf(N)$ for all $N > M$.*
- *· Ω(f(N)) means the set of all functions g(N) such that there exist positive constants C and M with*
  - *$| g(N) | > Cf(N)$ for all $N > M$.*
- *· Θ(f(N)) means the set of all functions g(N) such that there exist positive constants C, D and M with*
  - *$Cf(N) < | g(N) | < Df(N)$ for all $N > M$.*
-
- *Note. O( ) and Ω( ) are used to describe upper and lower bounds, Θ( ) we use for "optimal" algorithms. N is the measure of input size (number of points, bits, edges...).*

CompGeom Methodology

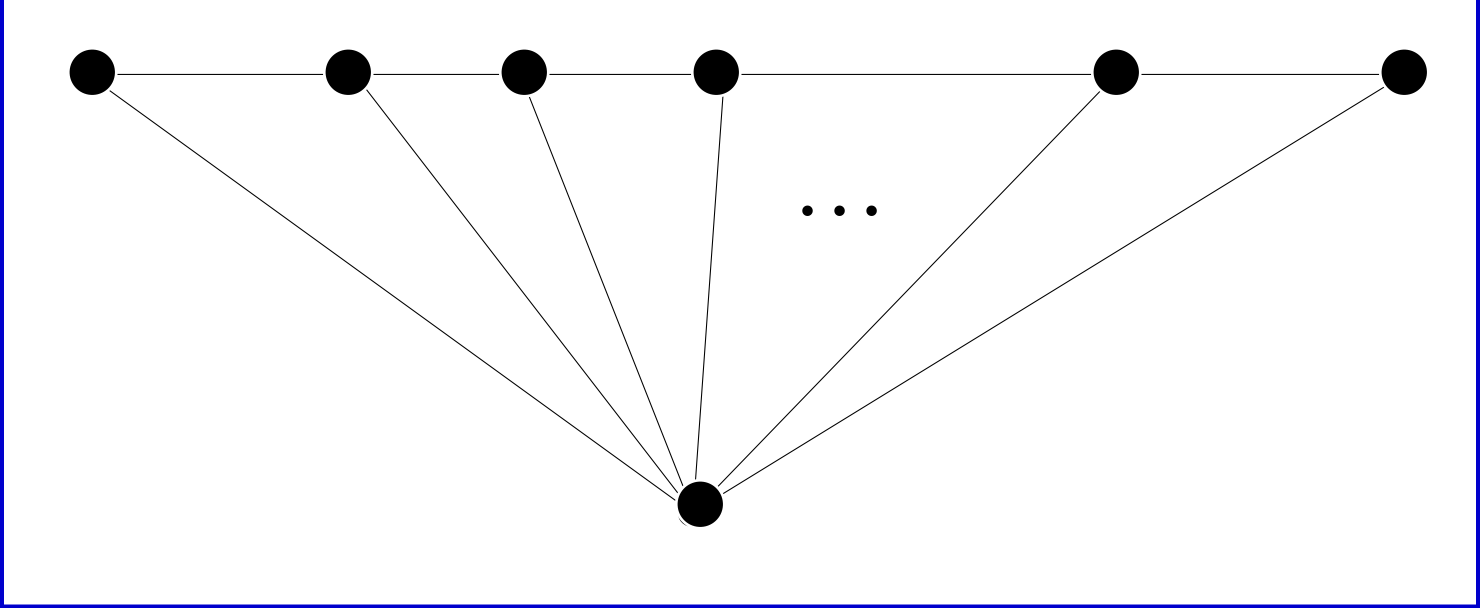
- *Lower bound – has to be proven, hard, usually by reduction to another known problem (e.g. sorting)*
- *Upper bound – any algorithm*



- *Efficient algorithm*
- *Optimal algorithm achieves the lower bound*
- *Complexity measures are time, memory, preprocessing time and memory, query time, (time of programming, output sensitivity, on-line and off-line problems… average complexity)*

CompGeom Methodology

- *Lower bound – has to be proven, hard, usually by reduction to another known problem (e.g. sorting)*



- *Triangulation sorts real numbers => $\Omega(N \log N)$*

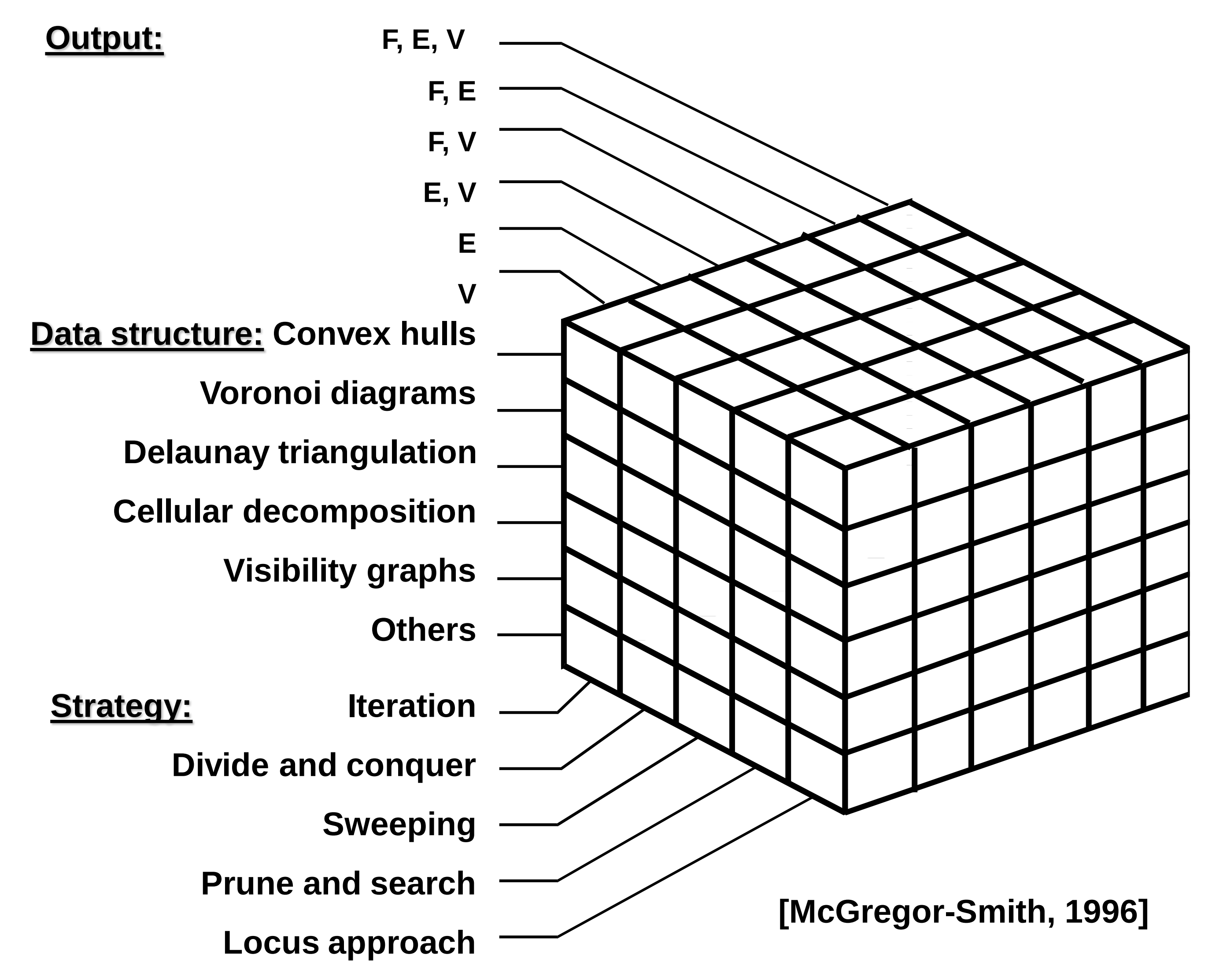
Algorithmic Strategies

- 1. Iteration
- 2. Sweeping
- 3. Sorting
- 4. Divide & Conquer
- 5. Locus Approach
- 6. Duality
- 7. Combinatorial Analysis

Algorithmic Strategies

- 8. Prune & Search
- 9. Dynamic Programming
- 10. ASA
- 11. Genetic Algorithms
- 12. Memetic Algorithms
- 13. DNA Computation, Neural Networks...
- 14. Darwish Camel, New Paradigms

CompGeom - 3 Ways to Explain



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  - HyperFun
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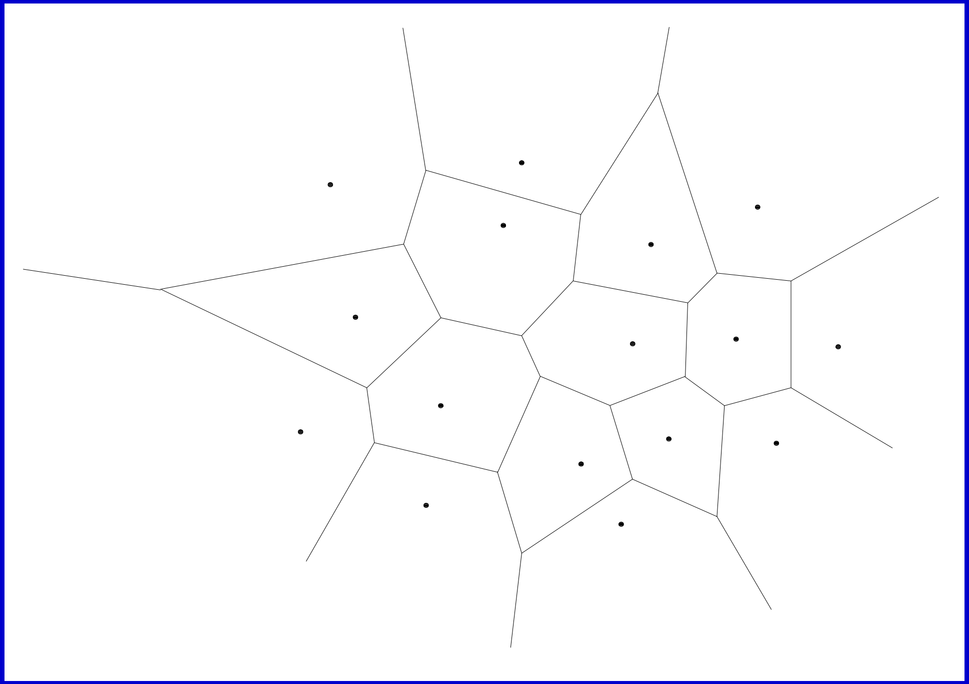
Voronoi Diagram – Many Names

- Model of both organic and anorganic originals: fences, lattices, spider web, soap bubbles, mineral crystals, honeycomb hexagons, tilings, Escherian space subdivisions… most fundamental geometric structure
- 

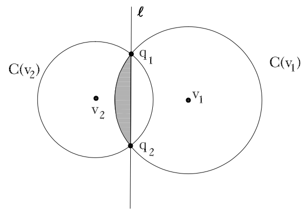
Voronoi Tesselation Problem

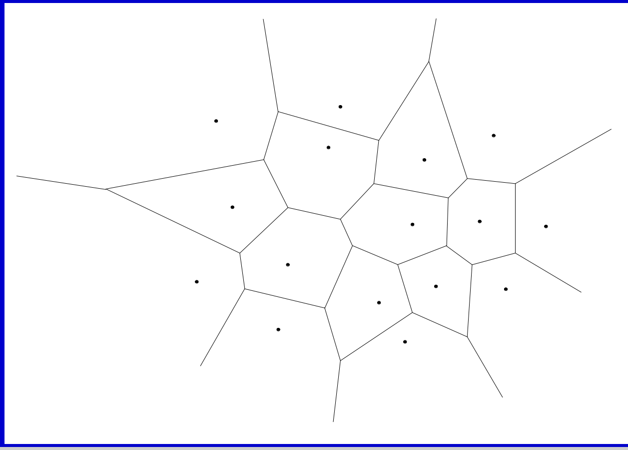
INPUT: Given a set P of N points in the plane, N finite, in general position

OUTPUT: Subdivide the plane according to proximity of points (regions of closest points)

REQUIREMENTS and MODIFICATIONS:

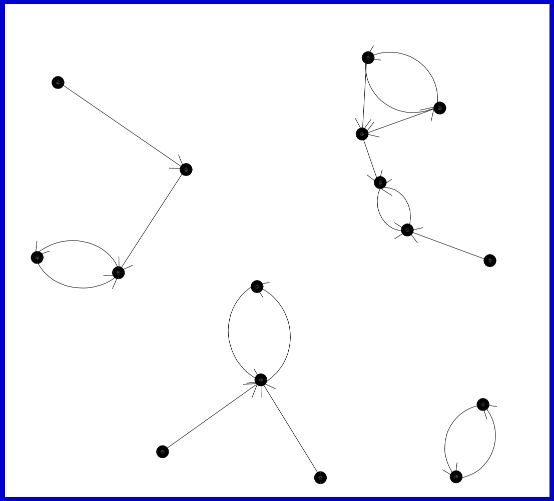
- **2D, 3D (no four cocircular)**
- **Various metrics**
- **Robotics (V-edges motion plan)**
- **Higher order**
- **Power diagrams**

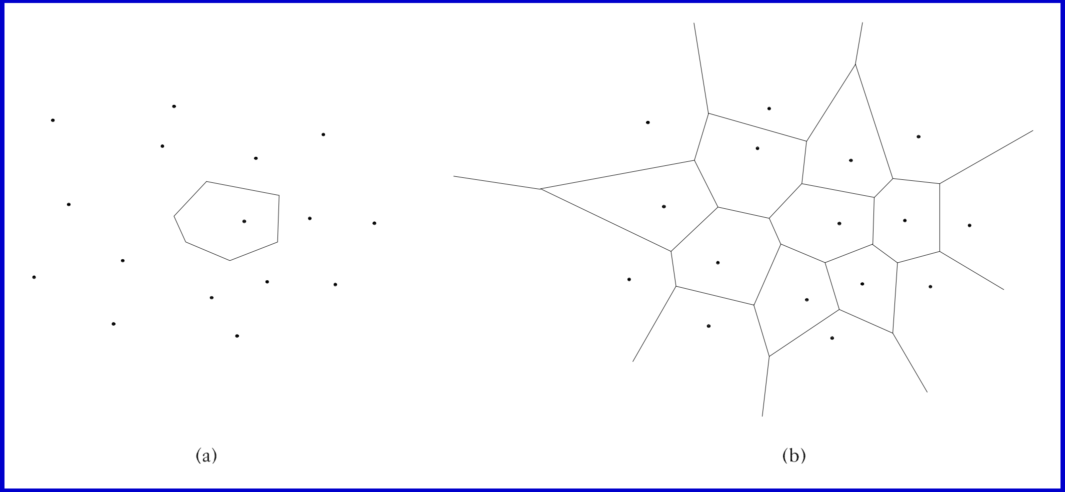




Related Problems

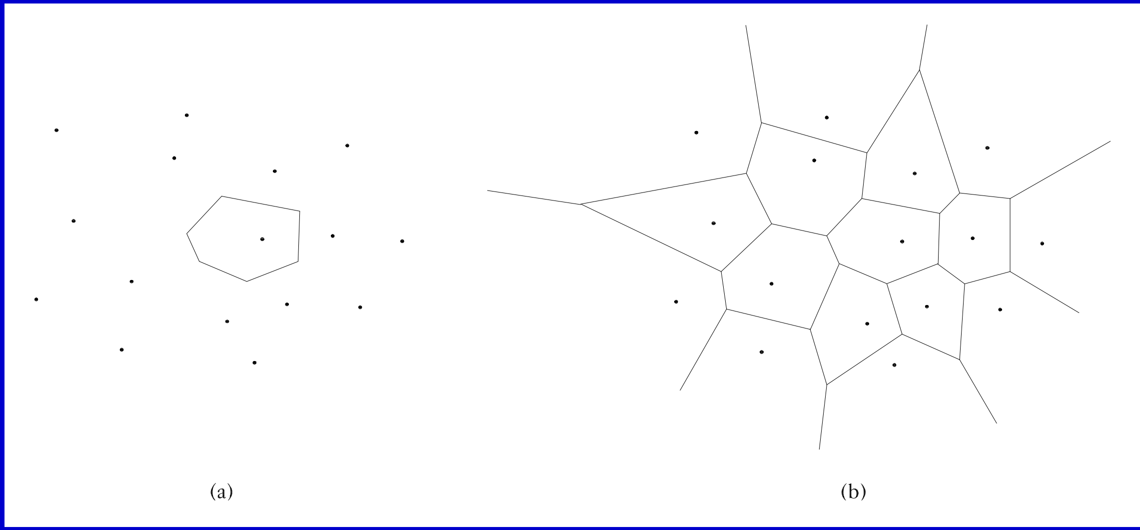
- Closest Pair
- All Nearest Neighbours, clusters
- Point Set Triangulation, paths
- Convex Hull
- Medial Axis, Blum
- TSP Heuristics
- …





Vor(P) Definitions

- *Voronoi diagram (Dirichlet tesselation) is a union of Voronoi polygons (tiling, no covering)*
- *Voronoi polygon, reg(p), is a locus of closest points, & a convex set, shares an edge with another one*
- *Voronoi point*
- *Generator*
- *Separator*



- *Specialised monographs: Okabe at al. 1997 and other*

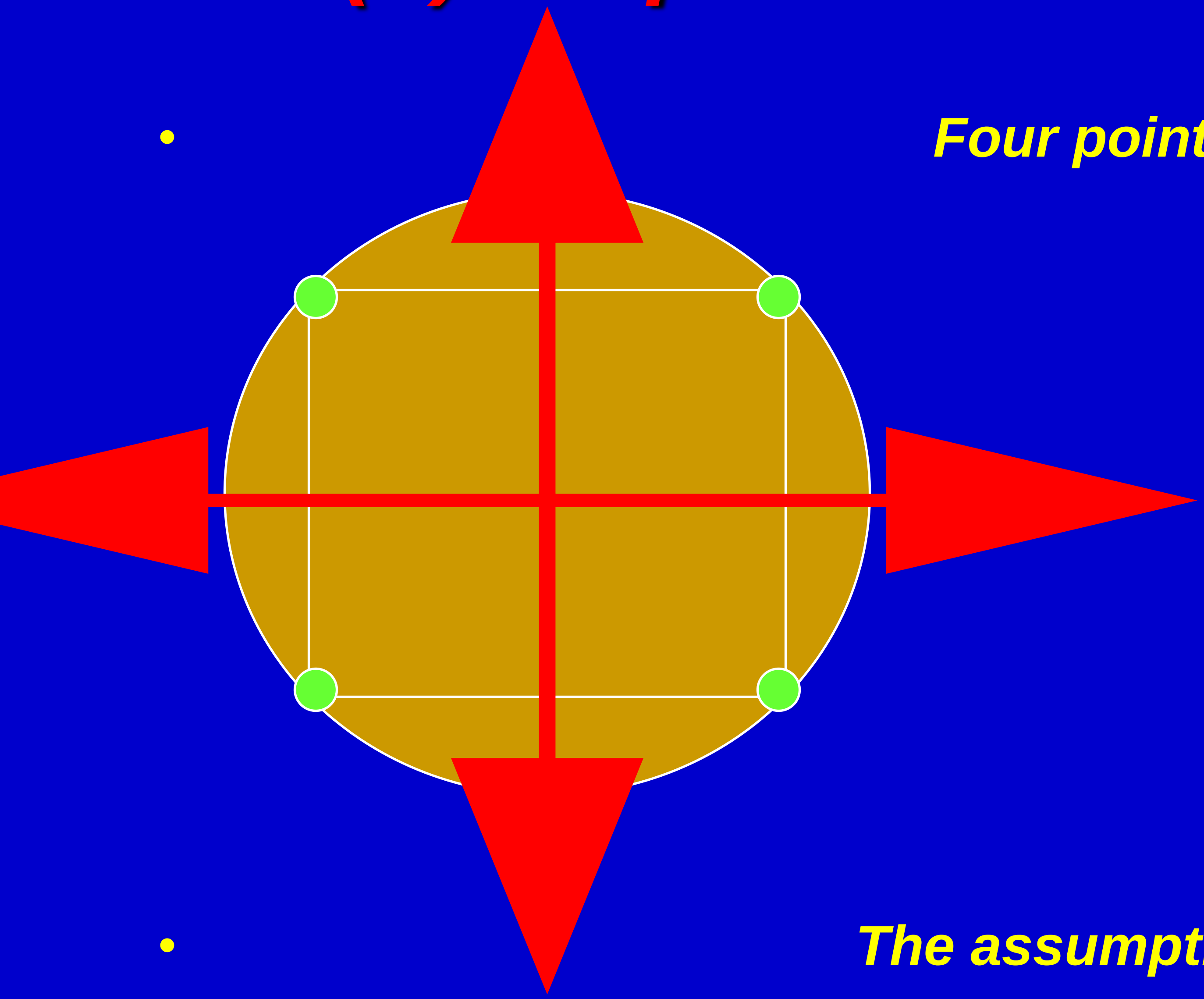
Vor(P) Properties

- Planar graph (Euler‘s formula) V - E + F = 2, everything LINEAR, O(N)

- N generators => N faces (some unbounded)
- No vertices for collinear input points
- Each vertex belongs to 3 edges and each edge has 2 vertices => 2E >= 3V, E <= 3N-6, V <= 2N-4
- Average number of edges for V-polygon is 5 or 6
- If a vertex p is closest to q then reg(p), reg(q) share an edge
- Each reg(p) is nonempty
- Unbounded regions contain extremal points

Vor(P) Properties for a Square

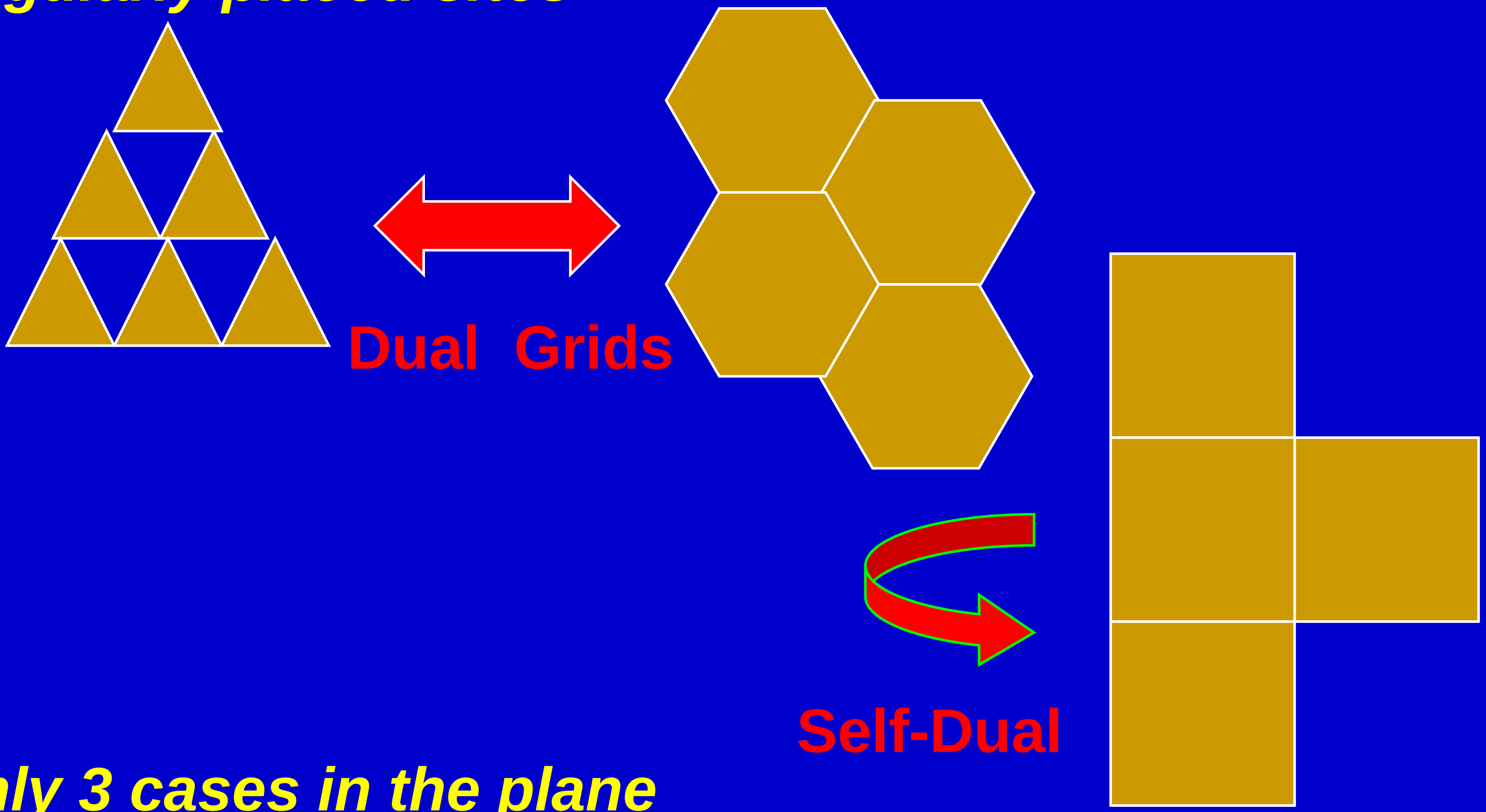
- Four points cocircular



- The assumption is not crucial

Vor(P) Special Cases, 2D

- *Regularly placed sites*
-



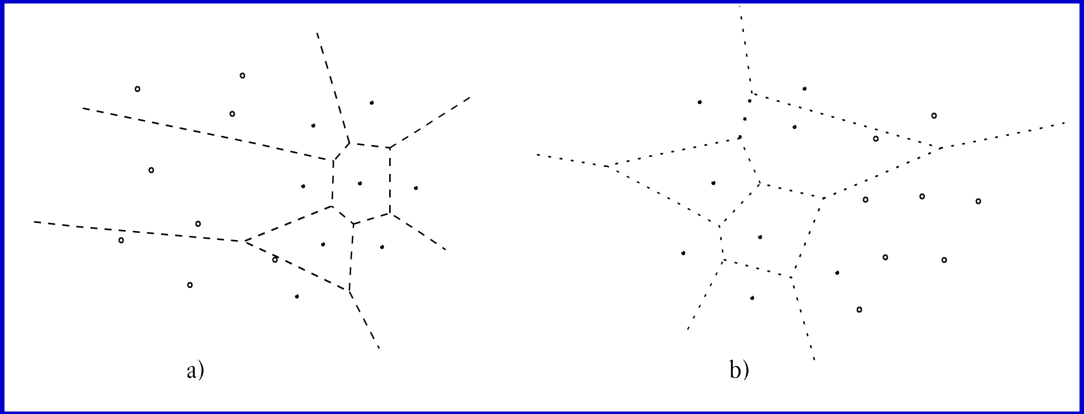
- *Only 3 cases in the plane*
- *Proof by integer division of 360 degrees*

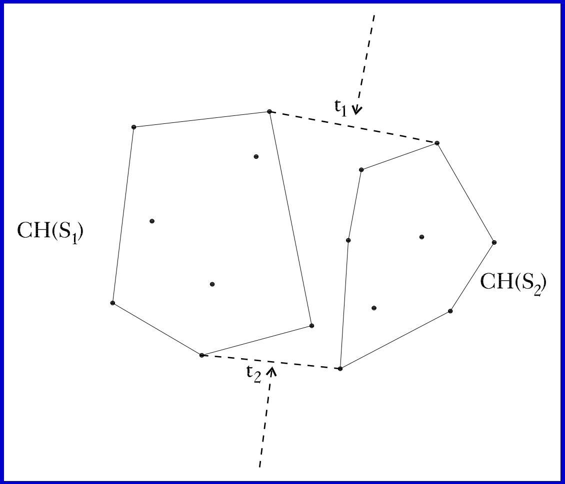
Vor(P) History

- *Voronoi, M. G. 1908. Nouvelles Applications des Parametres Continus a la Theorie des Formes Quadratiques. J. Reine Angew. Math. 134. Pp 198-287*
- *Gauss 1840 – quadratic forms (QF)*
- *Dirichlet 1850 – simple proof on irreducibility of QF*
- *Voronoi 1908 – generalization for d>2*
- *Thiessen 1911 - geography*
- *Horton 1917 – Thiessen polygons...*
- *Blum 1967 – new shape descriptors, Gestalt psychology*
- *... crystallography, databases, biology...*
- *Aurenhammer, ACM Surveys*
- *Okabe et al.*
-

Divide and Conquer Proof Sketch

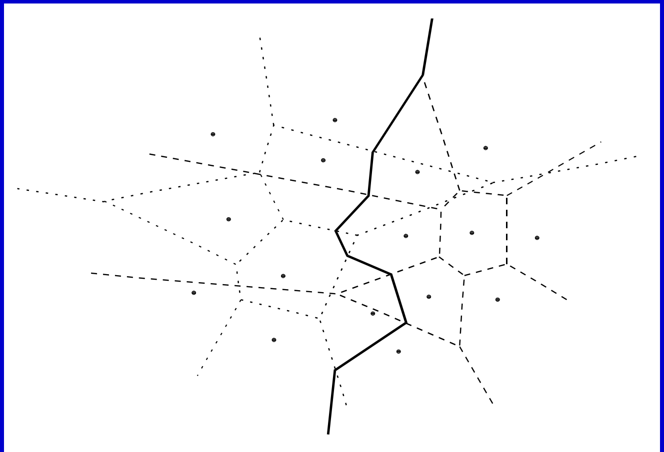
- Left and right parts





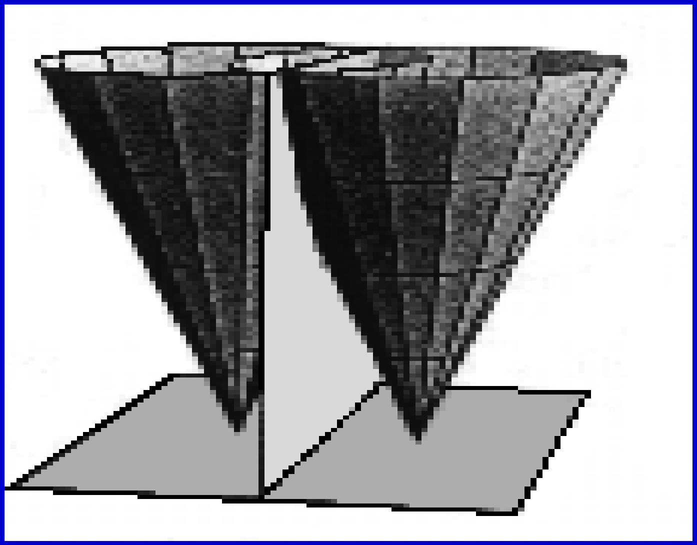
- Merge solutions

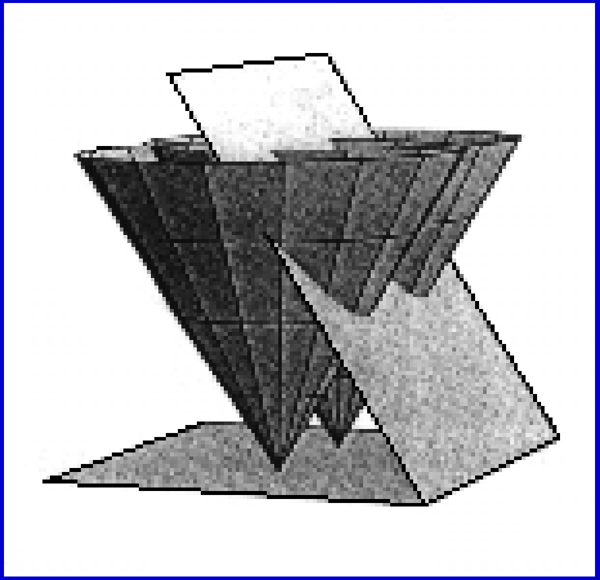
- O(NlogN)
- Optimal, but unstable
- Lower bound proof
- - too complex for today



Sweepline Construction

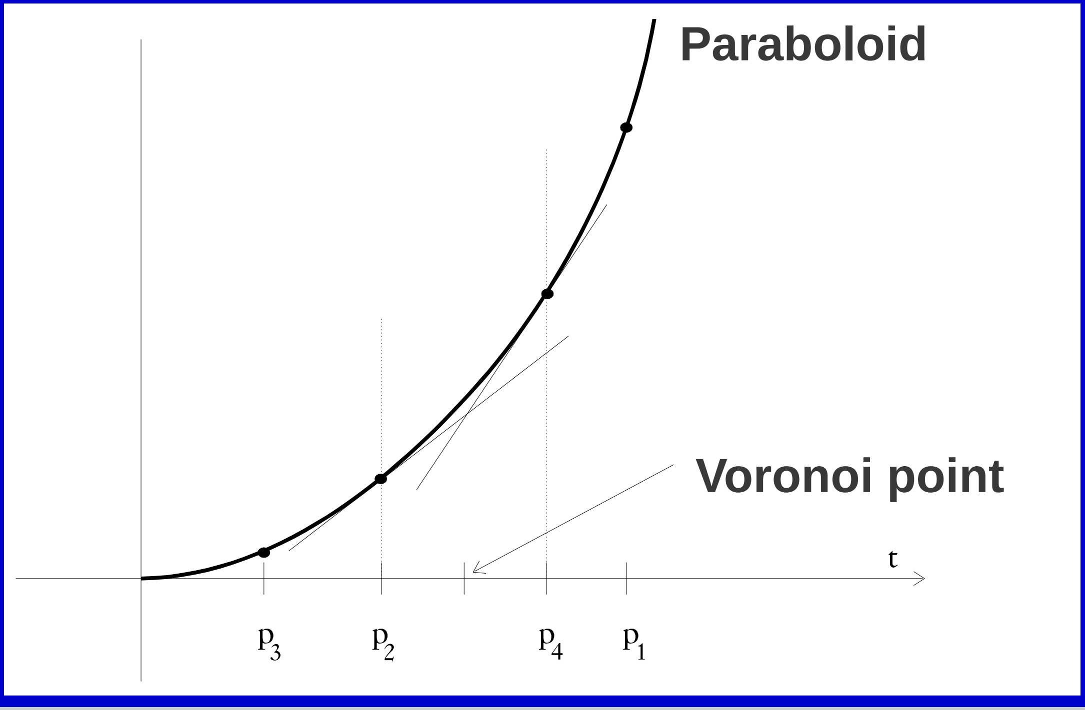
- *S. Fortune*
- *Geometric Interpretation by Guibas and Stolfi*





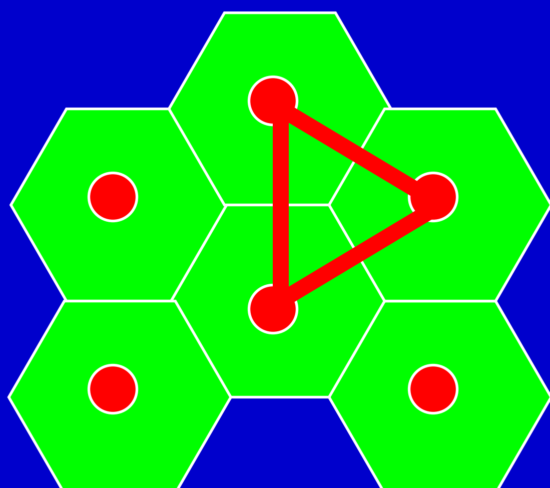
Lifting Transformation

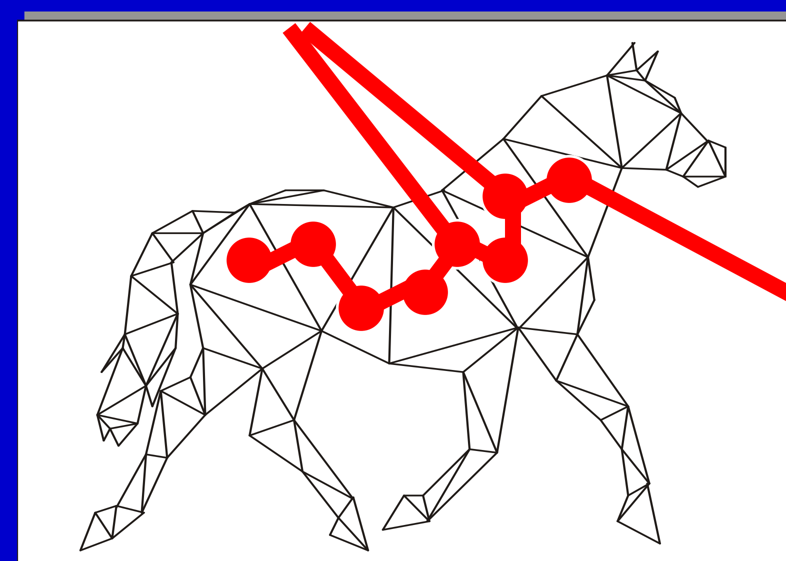
- 1. Elevate to paraboloid (linear time, O(N))
- 2. Compute convex hull (O(NlogN) even in 3D)
- 3. Return to the 2D plane (linear time, O(N))



Constructions using Triangulation

- Local optimality and global optimality
- Hint: Vor(P) is dual with some triangulation
- Prof. Aurenhammer, TU Graz, ACM Survey





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Triangulation Problem

INPUT: Given a set P of N points in the plane, N finite, in general position

OUTPUT: Subdivide the interior into O(N) non-overlapping triangles

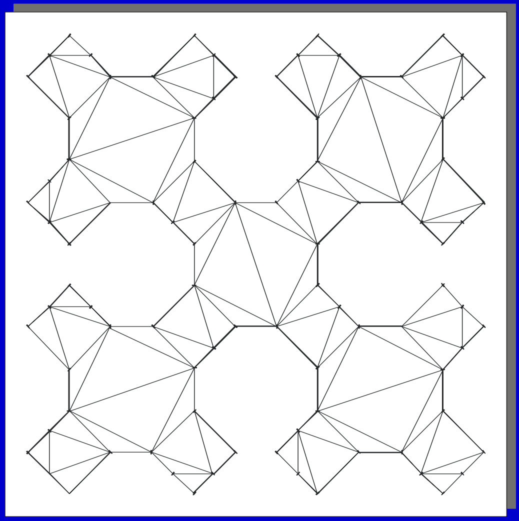
REQUIREMENTS and MODIFICATIONS:

- 2D, 2.5D, 3D (tetrahedralization)
- Maximal planar graph
- Triangles may share vertex/edge
- No Steiner points (mesh)

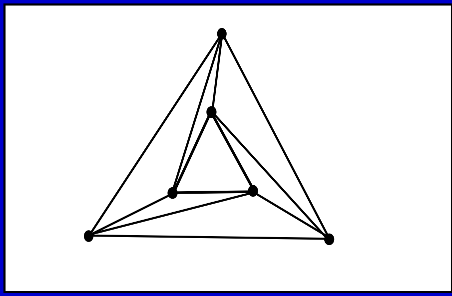
  Steiner points for mesh

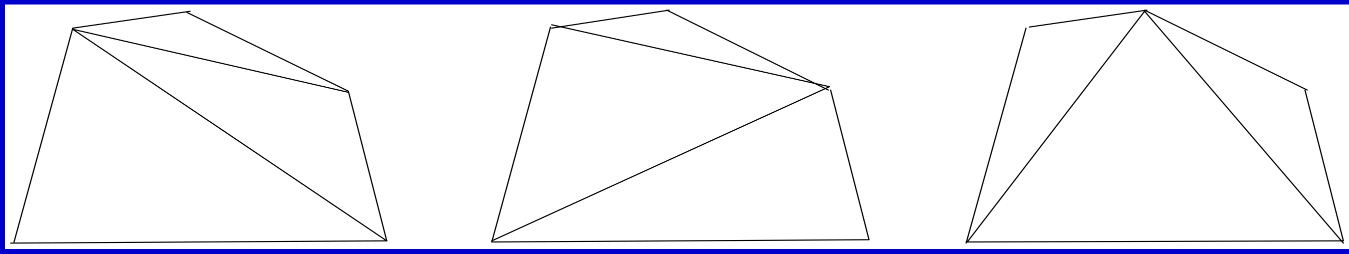
  Prescribed edges for constrained triangulation

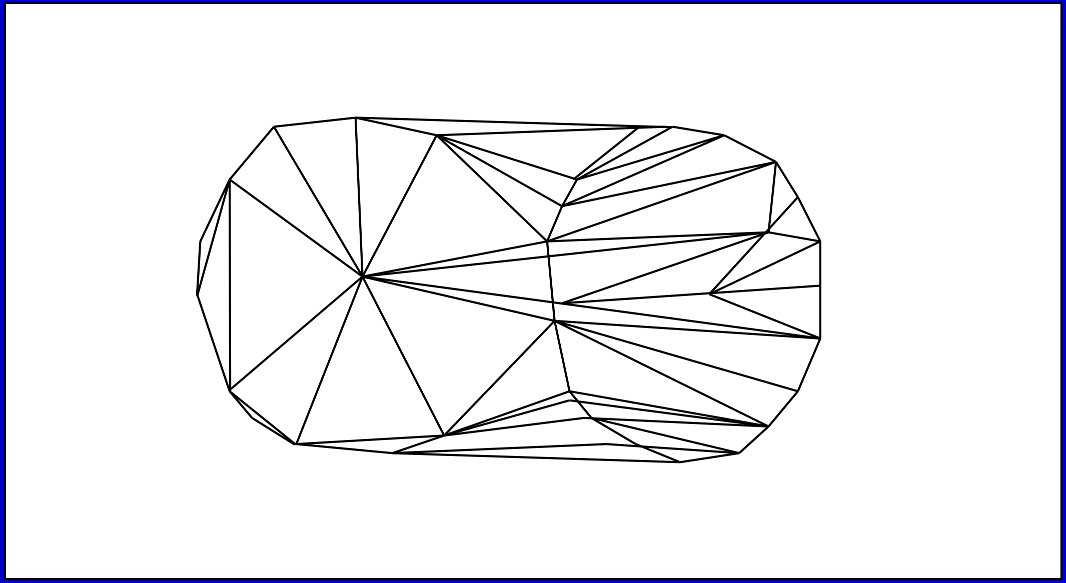
Planar Triangulations

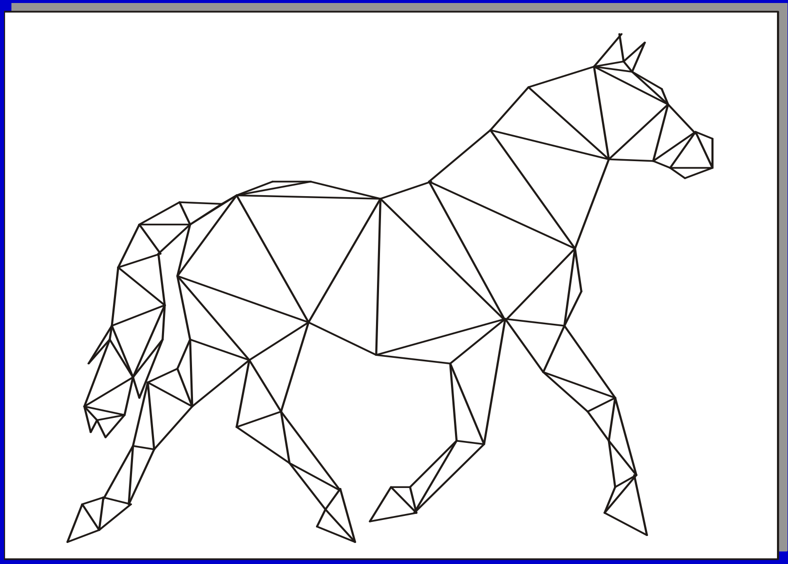


Optimisation criteria, triangle ordering, art gallery…



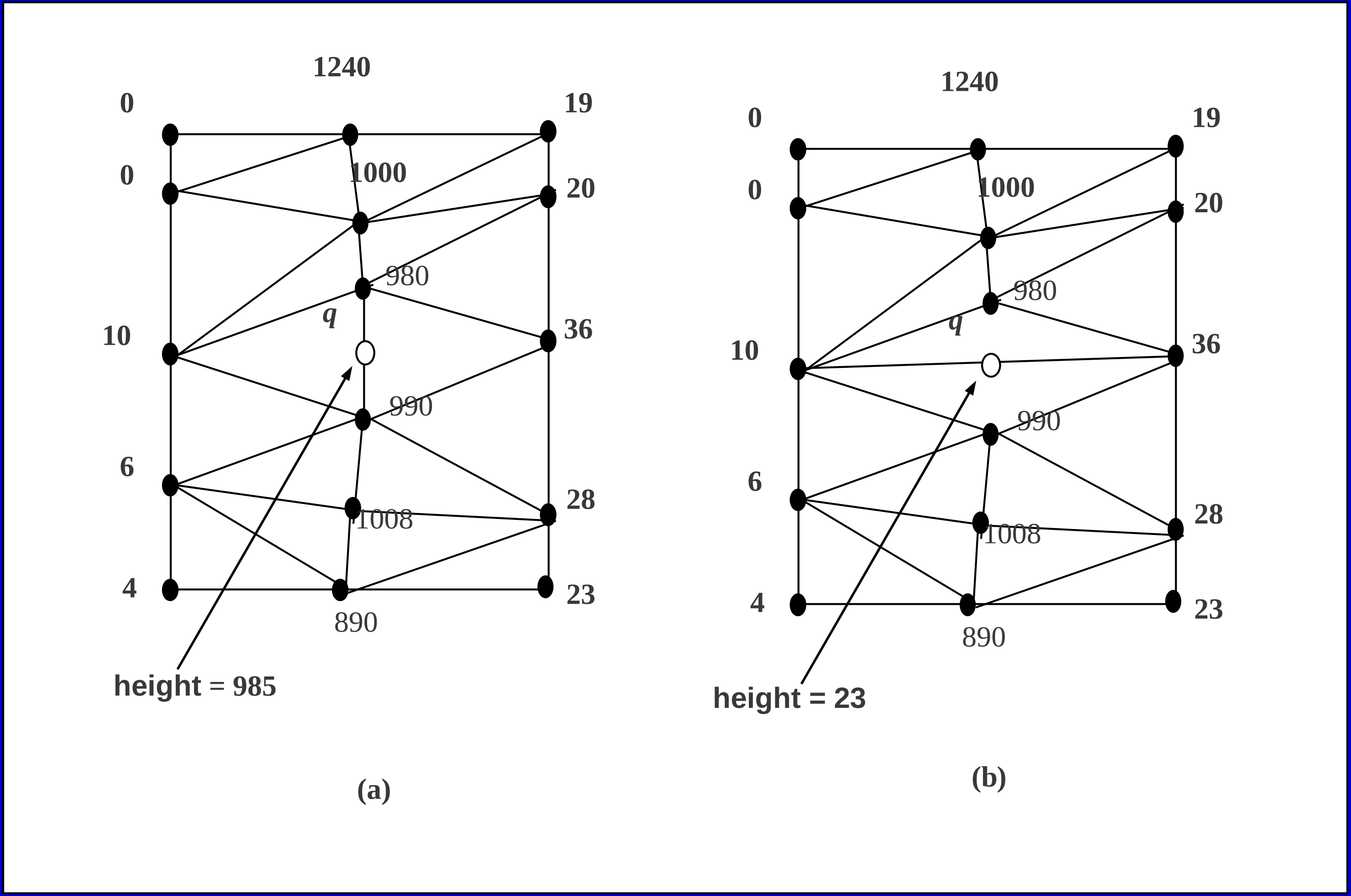






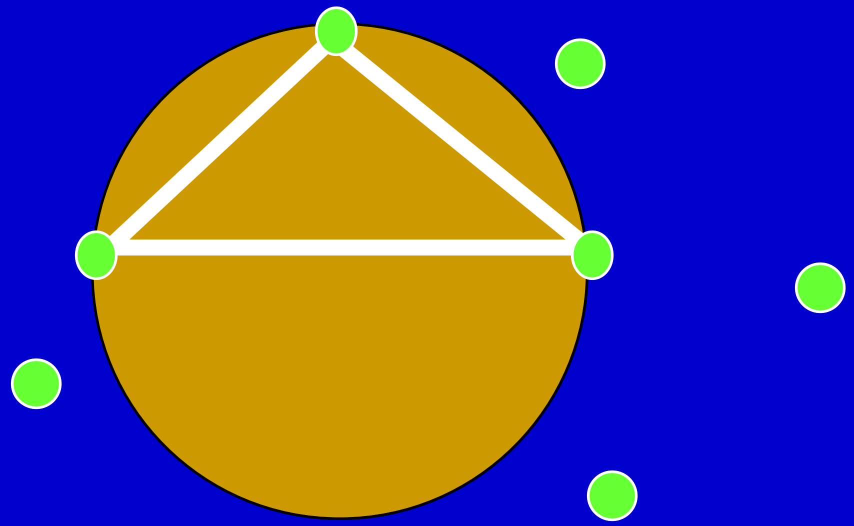
Terrain Interpolation

Approximation, minimum roughness property… (de Berg et al.)



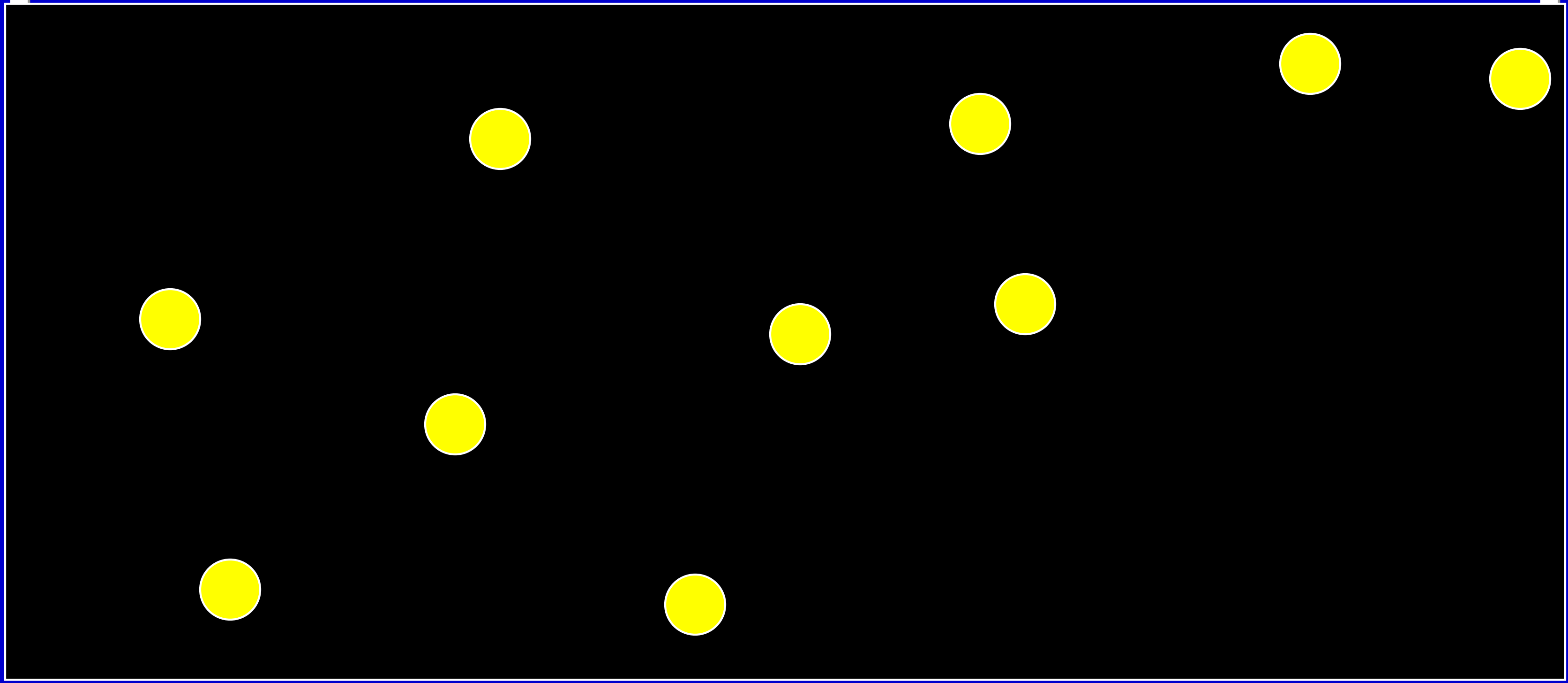
Lower bound and Optimal algorithm

Lower bound of triangulation problem:
O(NlogN) by reduction to sorting
achieved by Delaunay triangulation DT(P)

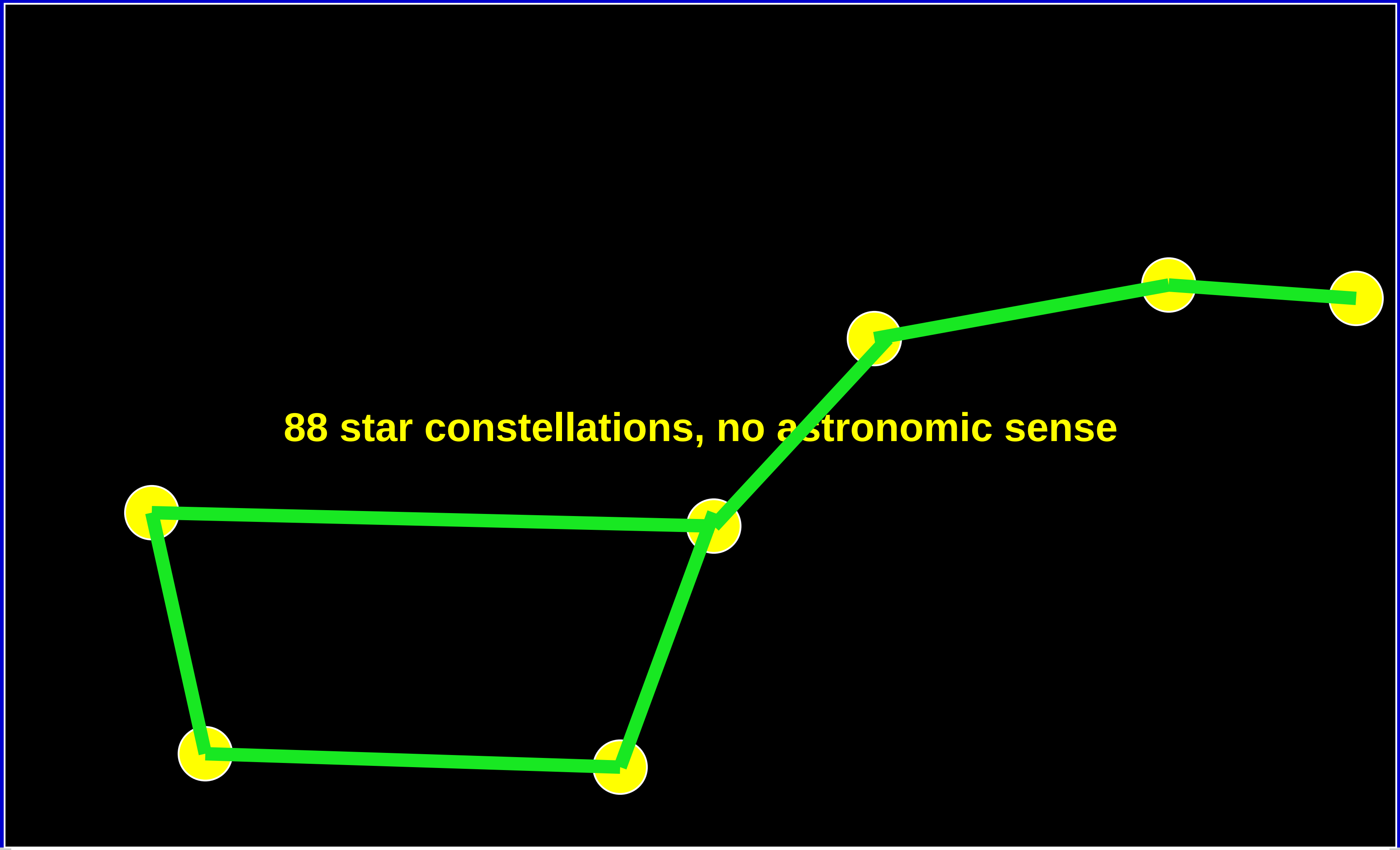


Point Constellations

Aurenhammer (2001): <u>14 309 547</u> sets of 10 points with respect to the different crossing properties



Star Constellations



DT(P) History

- Voronoi, M. G. 1908. Nouvelles Applications des Parametres Continus a la Theorie des Formes Quadratiques. J. Reine Angew. Math. 134. Pp 198-287
- Delaunay B. 1932. – student of Voronoi
- Numerical Interpolation and Finite Elements
- Wrong claim that DT(P) is optimal
- Delaunay refinement in mesh generation
- ...
- Aurenhammer, ACM Surveys
- Okabe et al.
- Bern, M. – Eppstein, B. Mesh Generation and Optimal Triangulations, a chapter in Computing in Euclidean Geometry

DT(P) Properties

- Graph theoretic dual to Vor(P)
- Planar graph with prominent subgraphs
- Cheapest triangulation
- Optimizes several criteria concerning triangle quality
- Extremely popular
- Alpha shapes, generalization of convex hull (non-convex)
- Beta skeletons

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Further Reading – Selected Books

- BOISSONNAT, J-D. - YVINEC, M. 1998. *Algorithmic Geometry.* 519 p. Cambridge: Cambridge University Press 1998. ISBN 0-521-56529-4.
- DE BERG, M. et al. 1997. *Computational Geometry, Algorithms and Applications.* 365 p. Berlin: Springer-Verlag.
- EDELSBRUNNER, H. 1987. *Algorithms in Combinatorial Geometry.* 423 p. Berlin: Springer-Verlag.
- GOODMAN, J. E. - O'ROURKE, J., eds. 1997. *Handbook of Discrete and Computational Geometry.* Boca Raton - New York: CRC Press.
- O'ROURKE, J. 1994. *Computational Geometry in C.* Cambridge: Cambridge University Press. 346 p.
- PREPARATA, F. P. - SHAMOS, M. I. 1985. *Computational Geometry: An Introduction.* 390 p. New York: Springer-Verlag.

- CHALMOVIANSKY, P. et al. 2001. *Zložitosť geometrických algoritmov*. UK.

Conclusions

- Motivation: optimal modeling
- Computational Geometry … optimal…
- Functional Representation … modeling…
- ... even multimedia objects (Pasko et al.)

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