Príklad 1:

$$C = \begin{pmatrix} 1 & 2 & c-1 \\ c-2 & 1 & 0 \\ c & 1 & 0 \end{pmatrix}, \quad c \in \mathbb{R}$$

- pre určenie regulárnosti / singulárnosti C rátame $\det(C)$:

$$\det C = \det \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{pmatrix} 1 & 2 & \boxed{c-1} \\ c-2 & 1 & \boxed{0} \\ c & 1 & \boxed{0} \end{pmatrix} \end{array} \overset{\text{rozvoj podľa 3. stĺpca}}{=} +(c-1)\begin{vmatrix} c-2 & 1 \\ c & 1 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & 2 \\ c & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} 1 & 2 \\ c-2 & 1 \end{vmatrix} =$$

$$= (c-1)\big(\underbrace{(c-2)\cdot 1 - c \cdot 1}_{c-2-c}\big) = (c-1)(-2)$$

- zaujíma nás, kedy $\det C = 0$:

C je REGULÁRNA $\iff \det(C) = (-2)(c-1) \neq 0 \iff c \neq 1$ $\boxed{c \in \mathbb{R} \setminus \{1\}}$

C je SINGULÁRNA $\iff \det(C) = (-2)(c-1) = 0 \iff c = 1$

- a teda lineárne zobrazenie $f_c : \mathbb{R}^3 \to \mathbb{R}^3$ s maticou C je LIN. IZOMORFIZMUS práve vtedy, keď $c \in \mathbb{R} \setminus \{1\}$.
(bijektívne, existuje f_c^{-1} s maticou C^{-1})

- ako by vyzeral predpis f_c:

$$f_c(\vec{x}) = (x_1, x_2, x_3) \cdot C = (x_1, x_2, x_3) \begin{pmatrix} 1 & 2 & c-1 \\ c-2 & 1 & 0 \\ c & 1 & 0 \end{pmatrix} =$$

$$= \big(x_1 + x_2(c-2) + x_3 \cdot c,\ 2x_1 + x_2 + x_3,\ x_1(c-1)\big)$$

čiže napr. pre $c = 3$ by sme mali $C = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \\ 3 & 1 & 0 \end{pmatrix}$

a lin. zobraz. $f_c : \mathbb{R}^3 \to \mathbb{R}^3$

$$f_c(x_1, x_2, x_3) = (x_1 + x_2 + 3x_3,\ 2x_1 + x_2 + x_3,\ 2x_1)$$

Príklad 2: $M = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$ $\quad M^{-1} = ?$ $\quad$ nad $\mathbb{R}$

- najprv zistíme, či M je regulárna (tj. či vôbec existuje M^{-1})

$$\det \begin{pmatrix} \boxed{\overset{+}{1}\ \overset{-}{1}\ \overset{+}{1}\ \overset{-}{0}} \\ 1\ 1\ 0\ 1 \\ 1\ 0\ 1\ 1 \\ 0\ 1\ 1\ 1 \end{pmatrix} = +1 \cdot \left|\begin{array}{ccc|cc} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{array}\right| - 1 \cdot \left|\begin{array}{ccc|cc} 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right| + 1 \cdot \left|\begin{array}{ccc|cc} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right| - 0 \cdot |\cdots|$$

$$= 1 \cdot (1.1.1 + 0.1.1. + 1.0.1 - 1.1.1 - 1.1.1 - 1.0.0) - 1.(1) + 1(-1) =$$

$$= 1.(-1) - 1 - 1 = \underline{\underline{-3}} \neq 0 \checkmark$$ M-regulárna

- hľadáme inverznú maticu $M^{-1} = (\det M)^{-1} \cdot \operatorname{adj} M$

$$(\det M)^{-1} = (-3)^{-1} \overset{\text{nad } \mathbb{R}}{=} -\frac{1}{3}$$ $\quad$ (lebo $(-3) \cdot (-\frac{1}{3}) = 1$ (neutr. prvok))

$$\operatorname{adj} M = \begin{pmatrix} M_{11} & M_{21} & M_{31} & M_{41} \\ M_{12} & M_{22} & M_{32} & M_{42} \\ M_{13} & M_{23} & M_{33} & M_{43} \\ M_{14} & M_{24} & M_{34} & M_{44} \end{pmatrix}$$

M_{11} z M: $\begin{pmatrix} \overset{\oplus}{1} & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \Rightarrow M_{11} = \oplus \det \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \underline{-1}$

M_{21} z M: $\begin{pmatrix} \overset{+}{1} & \overset{-}{1} & 1 & 0 \\ 1 & \oplus 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \Rightarrow M_{21} = \oplus \det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \underline{-1}$

M_{41} z M: $\begin{pmatrix} \overset{+}{1} & 1 & 1 & 0 \\ \overset{-}{1} & 1 & 0 & 1 \\ \overset{+}{1} & 0 & 1 & 1 \\ \ominus 0 & 1 & 1 & 1 \end{pmatrix} \Rightarrow M_{41} = -\det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = -(-2) = \underline{2}$

- dopočítame aj ostatné M_{ij} (v Maxime: adjoint(M);)

$$\operatorname{adj}(M) = \begin{pmatrix} -1 & -1 & -1 & 2 \\ -1 & -1 & 2 & -1 \\ -1 & 2 & -1 & -1 \\ 2 & -1 & -1 & -1 \end{pmatrix}$$

- potom inverzná matica M^{-1}:

$$M^{-1} = -\frac{1}{3} \begin{pmatrix} -1 & -1 & -1 & 2 \\ -1 & -1 & 2 & -1 \\ -1 & 2 & -1 & -1 \\ 2 & -1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 & 1/3 & -2/3 \\ 1/3 & 1/3 & -2/3 & 1/3 \\ 1/3 & -2/3 & 1/3 & 1/3 \\ -2/3 & 1/3 & 1/3 & 1/3 \end{pmatrix}$$ $\quad$ (v Maxime: invert(M);)

Príklad 3: nad $\mathbb{R}$: $\quad x_1 + 5x_2 + 4x_3 + 3x_4 = 1$ $\qquad (AX^T = B)$

$\qquad\qquad\qquad\qquad 2x_1 - x_2 + 2x_3 - x_4 = 0$

- aby sme mohli použiť Cramerovo pravidlo, potrebujeme „štvorcovú" maticu A, tj. na 4 neznáme 4 rovnice

- tým, že máme menej rovníc $^{(2)}$ ako neznámych $^{(4)}$, je jasné, že minimálne 2 neznáme budú parametre

- zvoľme $x_3 = s$, $x_4 = t$, kde $s, t \in \mathbb{R}$

- potom ekvivalentný systém:

$$\left.\begin{aligned} x_1 + 5x_2 + 4x_3 + 3x_4 &= 1 \\ 2x_1 - x_2 + 2x_3 - x_4 &= 0 \\ x_3 &= s \\ x_4 &= t \end{aligned}\right\} \text{4 rovnice pre 4 neznáme}$$

- čiže:

$$\det A = \det \begin{pmatrix} 1 & 5 & 4 & 3 \\ 2 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 \\ \boxed{0 \; 0 \; 0 \; 1} \end{pmatrix} = 1 \cdot \det \begin{pmatrix} 1 & 5 & 4 \\ 2 & -1 & 2 \\ \boxed{0 \; 0 \; 1} \end{pmatrix} = 1 \cdot \det \begin{pmatrix} 1 & 5 \\ 2 & -1 \end{pmatrix} = -11$$

$$D_1 = \det \begin{pmatrix} 1 & 5 & 4 & 3 \\ 0 & -1 & 2 & -1 \\ s & 0 & 1 & 0 \\ t & 0 & 0 & 1 \end{pmatrix} = -2t + 14s - 1$$

$$D_2 = \det \begin{pmatrix} 1 & 1 & 4 & 3 \\ 2 & 0 & 2 & -1 \\ 0 & s & 1 & 0 \\ 0 & t & 0 & 1 \end{pmatrix} = 7t + 6s - 2$$

$$D_3 = \det \begin{pmatrix} 1 & 5 & 1 & 3 \\ 2 & -1 & 0 & -1 \\ 0 & 0 & s & 0 \\ 0 & 0 & t & 1 \end{pmatrix} = -11s$$

$$D_4 = \det \begin{pmatrix} 1 & 5 & 4 & 1 \\ 2 & -1 & 2 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & t \end{pmatrix} = -11t$$

- použijeme Cramerove formuly: $\boxed{x_i = (\det(A))^{-1} \cdot D_i}$

$$x_1 = \left(-\frac{1}{11}\right) \cdot D_1 = -\frac{1}{11}(-2t + 14s - 1) = \frac{1}{11} - \frac{14}{11}s + \frac{2}{11}t$$

$$x_2 = \left(-\frac{1}{11}\right) \cdot D_2 = -\frac{1}{11}(7t + 6s - 2) = \frac{2}{11} - \frac{6}{11}s - \frac{7}{11}t$$

$$x_3 = \left(-\frac{1}{11}\right) \cdot D_3 = -\frac{1}{11}(-11s) = s$$ – KONTROLA

$$x_4 = \left(-\frac{1}{11}\right) D_4 = -\frac{1}{11}(-11t) = t$$

- záver: riešením nehomogénneho systému rovníc je množina $S_N = \left\{ \left(\frac{1}{11} - \frac{14}{11}s + \frac{2}{11}t, \frac{2}{11} - \frac{6}{11}s - \frac{7}{11}t, s, t\right) \in \mathbb{R}^4, s, t \in \mathbb{R} \right\}$

Príklad 4: matica $n \times n$ nad $\mathbb{R}$

$$D_n = \det\begin{pmatrix} 2 & 1 & 0 & 0 & \cdots & & & 0 \\ 1 & 2 & 1 & 0 & \cdots & & & 0 \\ 0 & 1 & 2 & 1 & \cdots & & & 0 \\ \vdots & & & & & & & \vdots \\ & & & \cdots & 1 & 2 & 1 & 0 \\ 0 & \cdots & & \cdots & 0 & 1 & 2 & 1 \\ 0 & \cdots & & \cdots & 0 & 0 & 1 & 2 \end{pmatrix} = ?$$

- vypočítajme prvých pár $D_1, D_2, \dots$

$D_1 = \det(2) = 2$

$D_2 = \det\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = 4 - 1 = 3$

$$D_3 = \det\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{matrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \end{matrix} = 8+0+0-0-2-2 = 4$$

$$D_4 = \det\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} = 5$$

$\vdots$

hypotéza: $\boxed{D_n = n+1}$

- počítajme:

$$D_n = \det\begin{pmatrix} \boxed{2\ 1\ 0\ 0\ \cdots\ 0} \\ 1\ 2\ 1\ 0\ \cdots\ 0 \\ 0\ 1\ 2\ 1\ 0\ \cdots\ 0 \\ 0\ 0\ 1\ 2\ 1 \\ \\ 0\ \cdots\ 0\ 1\ 2\ 1 \\ 0\ \cdots\ \cdots\ 0\ 1\ 2 \end{pmatrix} = 2\underbrace{\det\begin{pmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 1 & 0 \cdots & 0 \\ & & \ddots & & \\ & & 0 & 1 & 2 & 1 \\ & & & 0 & 1 & 2 \end{pmatrix}}_{D_{n-1}} - 1.\det\underbrace{\begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 2 & 1 & \cdots & 0 \\ 0 & & & & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & & & & & 2 \end{pmatrix}}_{(n-1)\times(n-1)}$$

$$= 2 \cdot D_{n-1} - 1.1.\det\underbrace{\begin{pmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 1 & & \vdots \\ 0 & 1 & 2 & & \vdots \\ \vdots & & & & \\ 0 & \cdots & & \cdots & 2 \end{pmatrix}}_{D_{n-2}} = \boxed{2D_{n-1} - D_{n-2}}$$

REKURENTNÝ VZŤAH

- teda máme, že: $D_n = 2D_{n-1} - D_{n-2}$ (toto sme dokázali)
- hypotézu $D_n = n+1$ ukážeme MI:

$n=1$: $D_1 = 1+1 = 2 = \det(2)$ ✓

IP: nech platí pre n, tj: $D_n = n+1$

$n+1$: $D_{n+1} = 2D_n - D_{n-1} \overset{IP}{=} 2(n+1) - ((n-1)+1) = 2n+2-n =$
$= n+2 = (n+1)+1$