$\vec{a}$

① $S = [(2,1,-1,0),(4,3,1,2),(-1,0,2,1)] \subset V_4(\mathbb{R})$

- nájdeme skutočnú dimenziu S

$$M_S = \begin{pmatrix} 2 & 1 & -1 & 0 \\ 4 & 3 & 1 & 2 \\ -1 & 0 & 2 & 1 \end{pmatrix} \sim \left(\begin{array}{cccc|c} 2 & 1 & -1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & 1 & 0 \end{array}\right)$$

$$\sim \begin{pmatrix} 2 & 1 & -1 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} = \vec{v}_1 \\ = \vec{v}_2 \\ \end{array} \Rightarrow h(M_S) = 2 \Rightarrow \dim(M_S) = 2$$

- hľadáme $S^\perp$ kde $\dim(S^\perp) = 2$

$\vec{x} \perp \vec{v}_1 \quad \langle \vec{x}, \vec{v}_1 \rangle = 0$

$x \perp \vec{v}_2 \quad \langle \vec{x}, \vec{v}_2 \rangle = 0$

- maticu si ešte upravíme na redukovanú stupňovitú:

$$\begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Riešenie homogénneho systému:

$x_4 = t$

$x_3 = s \qquad s, t \in \mathbb{R}$

$x_2 = -3s - 2t$

$x_1 = 2s + t$

$S_H = \{(2s+t, -3s-2t, s, t), s, t \in \mathbb{R}\} = S^\perp$

Za s a t postupne dosadíme 1:

$$S^\perp = [\underset{\vec{a}_1}{\underset{\shortparallel}{(2,-3,1,0)}}, \underset{\vec{a}_2}{\underset{\shortparallel}{(1,-2,0,1)}}]$$

(2) $a = 2 \quad b = -1$

$$M_f = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & 3 \\ 6 & 2 & -4 \end{pmatrix}$$

- jadro $Ker(f)$ $\quad f(\vec{x}) = \vec{0} \quad \vec{x} \cdot M_f = \vec{0}$

$$(x_1, x_2, x_3) \cdot \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & 3 \\ 6 & 2 & -4 \end{pmatrix} = (0,0,0)$$

$$(\underbrace{2x_1 - x_2 + 6x_3}_{=0} ; \underbrace{x_1 + 2x_3}_{=0} , \underbrace{x_1 + 3x_2 - 4x_3}_{=0}) = (0,0,0)$$

- riešime homogény systém

$$A = \left(\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 3 & -4 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 0 & \frac{1}{2} & -1 & 0 \\ 0 & \frac{7}{2} & -7 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 0 & \frac{1}{2} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_3 = t$

$x_2 + -2x_3 = 0$

$x_2 = 2x_3 = 2t$

$x_1 + 2x_3 = 0 \Rightarrow x_1 = -2t$

(2.)

- množina riešení: $\{(-2t, 2t, t) ; t \in \mathbb{R}\}$

teda vektor $\vec{x} = (-2t, 2t, t) \in Ker(f)$

ak $t = 1$ $\vec{x} = (-2\cdot 1, 2\cdot 1, 1) = (-2, 2, 1)$

$Ker(f) = [(-2, 2, 1)]$ $\dim(Ker(f)) = 1$

- obraz $Im(f)$

$$M(f) = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & 3 \\ 6 & 2 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{7}{2} \\ 0 & -1 & -7 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 7 \\ 0 & 0 & 0 \end{pmatrix}$$

$h(M(f)) = 2 \Rightarrow \dim f(V) = 2$

a $f(V) = [(1, \frac{1}{2}, \frac{1}{2}), (0, 1, 7)] = Im(f)$

↳ sú LN teda sú bázou

ideme z $\mathbb{R}^3 \to \mathbb{R}^3$

$\dim(V) = 3$ a $\dim(Im(g)) = 2$ nie je f injektívne

$\dim(W) = 3$ a $\dim(Im(g)) = 2$ nie je f surjektívne

3.

$(1,1,1) \overset{f}{=} (-1,1)$ $\qquad \mathbb{R}^3 \to \mathbb{R}^2$

$(-2,-1,1) \overset{f}{=} (2,1)$

$(-1,2,0) \overset{f}{=} (0,2)$

- skúmajme, či sú pôvodné vektory LN

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -1 & 1 \\ -1 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -8 \end{pmatrix}$$

$h(A)=3$ sú LN sme v $\mathbb{R}^3$ teda existuje práve 1 lineárne zobrazenie

Hľadáme $M(f)$

$$\left(\begin{array}{ccc|cc} 1 & 1 & 1 & -1 & 1 \\ -2 & -1 & 1 & 2 & 1 \\ -1 & 2 & 0 & 0 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|cc} 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & 3 & 0 & 3 \\ 0 & 3 & 1 & -1 & 3 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|cc} 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & 3 & 0 & 3 \\ 0 & 0 & 1 & \frac{1}{8} & \frac{3}{4} \end{array}\right) \sim \left(\begin{array}{ccc|cc} 1 & 0 & -2 & -1 & -2 \\ 0 & 1 & 3 & 0 & 3 \\ 0 & 0 & 1 & \frac{1}{8} & \frac{3}{4} \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|cc} 1 & 0 & 0 & -\frac{3}{4} & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{3}{8} & \frac{3}{4} \\ 0 & 0 & 1 & \frac{1}{8} & \frac{3}{4} \end{array}\right) \Rightarrow M(f) = \begin{pmatrix} -\frac{3}{4} & -\frac{1}{2} \\ -\frac{3}{8} & \frac{3}{4} \\ \frac{1}{8} & \frac{3}{4} \end{pmatrix}$$

3

- predpis zobrazenia:

$$f(\vec{x}) = f(x_1, x_2, x_3) = (x_1, x_2, x_3)\, M_f = \left(-\frac{3}{4}x_1 - \frac{3}{8}x_2 + \frac{1}{8}x_3,\ -\frac{1}{2}x_1 + \frac{3}{5}x_2 + \frac{3}{4}x_3\right)$$

- dôkaz, že zobrazenie je lineárne

$$f(\vec{x}+\vec{y}) = f\left[(x_1, x_2, x_3) + (y_1, y_2, y_3)\right] = f(x_1+y_1, x_2+y_2, x_3+y_3)$$

$$= \left[-\frac{3}{4}(x_1+y_1) - \frac{3}{8}(x_2+y_2) + \frac{1}{8}(x_3+y_3),\ -\frac{1}{2}(x_1+y_1) + \frac{5}{4}(x_2+y_2) + \frac{5}{7}(x_3+y_3)\right]$$

$$= \left[-\frac{3}{4}x_1 - \frac{3}{4}y_1 - \frac{3}{8}x_2 - \frac{3}{8}y_2 + \frac{1}{8}x_3 + \frac{1}{8}y_3,\ -\frac{1}{2}x_1 - \frac{1}{2}y_1 + \frac{3}{4}x_2 + \frac{3}{4}y_2 + \frac{3}{4}x_3 + \frac{3}{4}y_3\right)$$

$$= \left(-\frac{3}{4}x_1 - \frac{3}{8}x_2 + \frac{1}{8}x_3,\ -\frac{1}{2}x_1 + \frac{3}{4}x_2 - \frac{3}{4}x_3\right) + \left(-\frac{3}{4}y_1 - \frac{3}{8}y_2 + \frac{1}{8}y_3,\ -\frac{1}{2}y_1 + \frac{3}{4}y_2 + \frac{3}{4}y_3\right)$$

$$= f(\vec{x}) + f(y)$$

$$f(\lambda\vec{x}) = f(\lambda(x_1, x_2, x_3)) = \left(-\frac{3\lambda}{4}x_1 - \frac{3\lambda}{8}x_2 + \frac{1\lambda}{8}x_3,\ -\frac{1\lambda}{2}x_1 + \frac{3\lambda}{4}x_2 + \frac{3\lambda}{4}x_3\right)$$

$$= \lambda\left(-\frac{3}{4}x_1 - \frac{3}{8}x_2 + \frac{1}{8}x_3,\ -\frac{1}{2}x_1 + \frac{3}{4}x_2 + \frac{3}{4}x_3\right) = \lambda \cdot f(\vec{x})$$

4) $\left[\overset{\vec{a}_1}{(1,0,2)}, \overset{\vec{a}_2}{(1,1,-3)}, \overset{\vec{a}_3}{(0,1,-1)}\right] = V_3(\mathbb{R})$

↳ keďže tieto vektory sú bázou $V_3(\mathbb{R})$ nemusíme kontrolovať či sú LN, lebo sú.

$$\vec{g}_1 = (1,0,2) = \vec{a}_1$$

$$\vec{g}_2 = \vec{a}_2 + \alpha \cdot \vec{g}_1 = (1,1,-3) + \alpha \cdot (1,0,2)$$

$$\alpha = -\frac{\langle \vec{a}_2, \vec{g}_1 \rangle}{\langle \vec{g}_1, \vec{g}_1 \rangle} = -\frac{1-6}{1+4} = -\frac{-5}{5} = 1$$

$$\vec{g}_2 = (1,1,-3) + 1 \cdot (1,0,2) = (2,1,-1)$$

$$\vec{g}_3 = \vec{a}_3 + \alpha \cdot \vec{g}_1 + \beta \cdot \vec{g}_2$$

$$\alpha = -\frac{\langle \vec{a}_3, \vec{g}_1 \rangle}{\langle \vec{g}_1, \vec{g}_1 \rangle} = -\frac{-2}{5} = \frac{2}{5}$$

$$\beta = -\frac{\langle \vec{a}_3, \vec{g}_2 \rangle}{\langle \vec{g}_2, \vec{g}_2 \rangle} = -\frac{1+1}{4+1+1} = \frac{-2}{6} = \frac{-1}{3}$$

$$\vec{g}_3 = (0,1,-1) + \frac{2}{5}(1,0,2) - \frac{1}{3}(2,1,-1)$$

$$= (0,1,-1) + \left(\frac{2}{5}, 0, \frac{4}{5}\right) + \left(\frac{-2}{3}, \frac{-1}{3}, \frac{1}{3}\right)$$

$$= \left(\frac{-4}{15}, \frac{2}{3}, \frac{2}{15}\right) / \cdot 15 = (-4, 10, 2)$$

- vektory $(1,0,2)$, $(2,1,-1)$, $(-4,10,2)$ sú navzájom na seba kolmé a LN teda generujú celý VP $V_3(\mathbb{R})$

5

$$A = \left(\begin{array}{cccc|c} 4 & 4 & 0 & 2 & 4 \\ 0 & -1 & -1 & 0 & -3 \\ 1 & 4 & 3 & 1 & 4 \\ 3 & 2 & -1 & 1 & 6 \end{array}\right) \sim \left(\begin{array}{cccc|c} 4 & 4 & 0 & 2 & 4 \\ 0 & -1 & -1 & 0 & -3 \\ 0 & 3 & 3 & \frac{1}{2} & 3 \\ 0 & -1 & -1 & -\frac{1}{2} & 3 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 4 & 4 & 0 & 2 & 4 \\ 0 & -1 & -1 & 0 & -3 \\ 0 & 0 & 0 & \frac{1}{2} & -6 \\ 0 & 0 & 0 & -\frac{1}{2} & 6 \end{array}\right) \sim \left(\begin{array}{cccc|c} 4 & 4 & 0 & 2 & 4 \\ 0 & -1 & -1 & 0 & -3 \\ 0 & 0 & 0 & \frac{1}{2} & -6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$h(A) = 3 \qquad h(A|B) = 3 \Rightarrow$ sústava má riešenie

- ešte si to upravíme na redukovaný stupňovitý tvar :

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -12 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -12 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$x_3 = t \qquad x_2 + t = 3 \Rightarrow x_2 = 3 - t$

$x_4 = -12 \qquad x_1 = 4 + t$

$$S_N = \{(4+t, 3-t, t, -12), t \in \mathbb{R}\}$$

$$S_N = \underbrace{(4, 3, 0, -12)}_{k} + \underbrace{\{(t, -t, t, 0), t \in \mathbb{R}\}}_{S_H}$$