QUADRATIC SPACE-LIKE BÉZIER CURVES IN THREE DIMENSIONAL MINKOWSKI SPACE

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Abstract. This paper consider about quadratic Bézier curves in three-dimensional Minkowski space. We shall show the conditions for the control points A, C, B of the Bézier curve such that the Bézier segment is space-like. For the middle control point C, we shall give a geometrical interpretation of the feasibility condition.

Keywords: Bézier curve, space-like curve, conic section.

1. Introduction

We will continue in the spirit of the works [3,4] where the set of admissible points C is described for conic sections such as unit circle and ellipse. In this paper, we show the set of admissible points for parabola. At first, we recall some facts about Minkowski space, more on this topic can be found in the books [1,2]. Then, we will show how the spatial problem can be reduce to the planar problem and we solve it for the special case of conic section.

2. Minkowski space, Bézier curves and their properties

Pseudo-euclidean space, denoted by $\mathbb{R}^n_p$, $n, p \in \mathbb{N}$ is an $n-$dimensional real vector space with the pseudo-scalar product $\langle \overline{x}, \overline{y} \rangle = \sum_{i=1}^{n-p} x_i y_i - \sum_{j=n-p+1}^{n} x_j y_j$ for $\overline{x} = (x_1, \ldots, x_n)$, $\overline{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n_p$. This inner product is a non-degenerate, symmetric bilinear form. For $p = 1$ we call it *Minkowski space*.

We call the vector $\overline{x} \in \mathbb{R}^n_1$ *space-like* if $\langle \overline{x}, \overline{x} \rangle > 0$, *time-like* if $\langle \overline{x}, \overline{x} \rangle < 0$ and *light-like* if $\langle \overline{x}, \overline{x} \rangle = 0$. The set of all light-like vectors forms the *light cone* or *isotropic cone* Q. Although we consider vector space, we could work also in an affine space with a *pseudo-cartesian* coordinate system $S(O, x_1, \ldots, x_n)$, where the axes $x_1, \ldots, x_n$ are pseudo-orthogonal and O is the origin. We say that the coordinate axes $x_1, \ldots, x_{n-p}$ are space-like and the axes $x_{n-p+1}, \ldots, x_n$ are time-like. We call a point $x \in \mathbb{R}^n_1$ *space-like* (*time-like*, *light-like* respectively) if its position vector $\overline{x} = \overline{Ox}$ is such a one. There are two possible ways, how to define space-like curve (time-like and light-like respectively). A curve $p : I \to \mathbb{R}^n_1$ is called space-like if the tangent vector $\dot{p}(t)$ is space-like for each $t \in I$. There is an alternative definition of the space-like curve. A *curve* $p(t)$ *is space-like* if it contains only space-like points, i.e. $\overline{p(t)} = \overline{Op(t)}$ is space-like vector for every $t \in I$. We will use the second definition.

Bézier curve in Minkowski space of degree n is the polynomial map $b : [0, 1] \to \mathbb{R}^n_1$

[1]This project has been supported by the project UK/279/2011.
[2]This project has been supported by the project VEGA 1/0730/09.

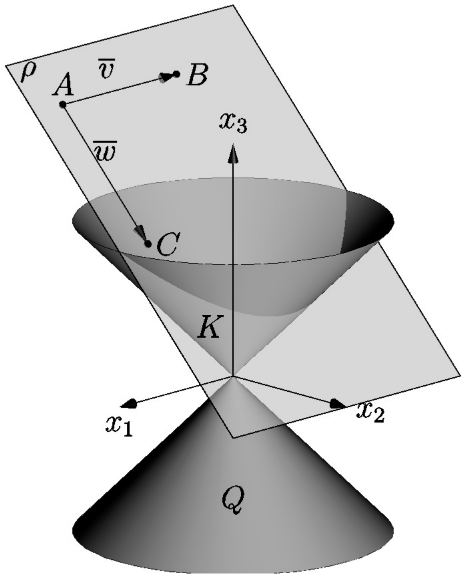


Fig. 1. The location of the plane ρ and consequently the shape of conic section depends on the position of the point C

that $b(t) = \sum_{i=0}^{n} B_i^n(t)\, b_i$ for $t \in [0,1]$. Points $b_i \in \mathbb{R}_1^n$ are called control points, $B_i^n(t) = \binom{n}{i}(1-t)^{n-i}t^i$ for $i \in \{0,\dots,n\}$ are Bernstein polynomials of degree n. The Bézier curve $b(t)$ always passes through the first and the last control point and lies within the convex hull of its control points. Bézier curves are invariant under affine transformations, but not invariant under all projective transformations.

3. Space-like conditions

Let us consider a Minkowski space $\mathbb{R}_1^3$ and a quadratic Bézier curve with control points A, C,B in this order, denoted by $b_{ACB}(t)$. In order a Bézier curve to be space-like, each of its points have to be space-like. Let the points $A = [a_1, a_2, a_3]$ and $B = [b_1, b_2, b_3]$ be fixed. Hence, we have necessary and sufficient conditions

$$a_1^2 + a_2^2 - a_3^2 > 0\,, \tag{1}$$
$$b_1^2 + b_2^2 - b_3^2 > 0\,. \tag{2}$$

We are looking for the set of all such points C that the Bézier curve $b_{ACB}(t)$ is space-like.

Since a quadratic Bézier curve is a part of a parabola, it lies in some plane ρ given by $\rho = A + t\overline{v} + s\overline{w}$ for $t, s \in \mathbb{R}$, where the vector $\overline{v} = \overline{AB}$ and an unknown vector $\overline{w} = \overline{AC}$. As the position of the point C changes, the plane ρ is rotates about the axis $\overleftrightarrow{AB}$. The intersection of the light-cone Q and the plane ρ is a conic section K (see fig. 1.).

The space-like Bézier curve $b_{ACB}(t)$ lies outside of the light-cone Q, hence outside of conic section K in the plane ρ. We solve it in the plane ρ for each type of conic section and the planar results can be complete to the spatial result.

Let $S_\rho(O, x, y)$ be a pseudo-cartesian coordinate system in the plane ρ. Let $A = [a_x, a_y]$, $C = [c_x, c_y]$ a $B = [b_x, b_y]$ be coordinates of the control points in $S_\rho(O, x, y)$. The conic section

$K = \{[x,y] \in \mathbb{R}^2 : k_A x^2 + 2k_B xy + k_C y^2 + 2k_D x + 2k_E y + k_F = 0\}$. In appropriate cases, we consider the equation of the conic section.

4. Domain of admissible middle control points

Let V be a set of points C such that the curve b_{ACB} is space-like. Then, we say that V is a *set of admissible solutions*.

We say that the set $D \subset K$ is the *set of points of contact* if for any point $X = [x_0, y_0] \in D$ there is a point C such that $b_{ACB}(t) \cap K = M$, where $X \in M$ is a point of contact of order 2 and the set M contains only points of contact of order 2. The set M contains at most two elements, since two different quadratic curves may have at most two common points of contact of order 2.

We say that $b_{ACB}(t)$ *touches* K from the *outside (inside)*, if $b_{ACB}(t)$ and K are (not) separated by their common tangent up to the point of contact. Then, the point of contact is called *exterior (interior) point of contact*. The set of all exterior (interior) points of contact is denoted D_{out} (D_{in}). The set of points of contact $D = D_{out} \cup D_{in}$ for the circle and ellipse K was described in the works [3] and [4].

The following propositions describes the set of points of contact for the parabola $K = \{[x,y] \in \mathbb{R}^2 : x^2 - y = 0\}$. From the space-like points A, B, we can construct tangent lines to K, two from each. Let us denote one of them t. The tangent line t divides the plane ρ into two half-planes H_t^+, H_t^-. We call the tangent line t *separating* (denoted by t_{sep}), if the conic section K and the segment AB lie in different half-planes with respect to the tangent t, i.e. $K \in H_t^+$ and $AB \in H_t^-$, see figure 3. .

Lemma 1. *Let A, B be two different space-like points, such that the line $\overleftrightarrow{AB}$ is not tangent to K. Then among all tangents from A and B to K, there are two separating.*

We can easily proove the lemma by analyzing the position of the point A in the areas numbered by 1,2,3,4 in the figure 2. .

Theorem 2 (Set of exterior points of contact)**.** *Let A, B be space-like such that line $\overleftrightarrow{AB}$ is not tangent to K. The set of exterior points of contact $D_{out} \neq \emptyset$ if and only if the segment $AB \cap K = \emptyset$. The set D_{out} consists of a continuous arc $\overset{\frown}{T_1T_2}$ on K, where points T_1, T_2 are points of contact for separating tangent lines t_{sep1}, t_{sep2} from the points A, B (see fig. 3.). The points $T_1, T_2 \notin D_{out}$.*

The theorem says that the set of the exterior points of contact is not empty for the points A, B such that line $\overleftrightarrow{AB}$ is not tangent to K if and only if the point A lies in one of the areas 1,2,3,4 in the figure 2. .

We say that $b_{ACB}(t)$ has *double contact*, if it has with K two points of contact of order 2. The line $\overleftrightarrow{AB}$ divides the plane ρ into two half-planes, the open half-plane H_{AB}^-, and the closed half-plane $\overline{H_{AB}^+}$ such that $\overleftrightarrow{AB} \subset H_{AB}^+$. Let us divide tangent lines from A, B into two pairs t_1^+, t_2^+ and t_1^-, t_2^- such that the relevant points of contact $T_i^{+,-} = \overline{t_i^{+,-}} \cap K, i = 1,2$ lie in the same half-plane $T_{1,2}^+ \in \overline{H_{AB}^+}$ and $T_{1,2}^- \in H_{AB}^-$. If $t_1^+ \cap t_2^+ = P^+ \in \overline{H_{AB}^+}$, then we say that tangents t_1^+, t_2^+ *converge*, see figure 3. . If $P^+ \in H_{AB}^-$, then we say they *diverge*. The same definition holds for the pair t_1^-, t_2^- and for separating (or not) tangent lines in the case of $AB \cap K = \emptyset$.

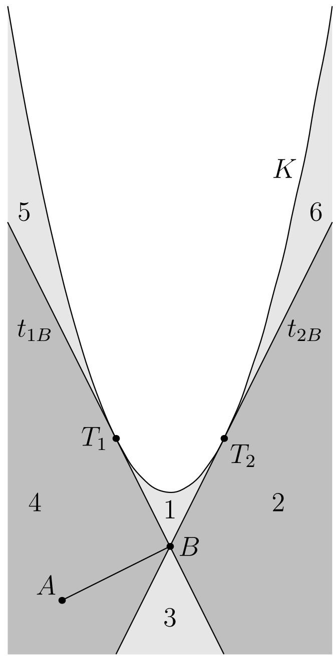


Fig. 2. If line $\overleftrightarrow{AB}$ is not tangent to the conic K, then space-like A can lie only in one of the areas 1,...,6 designated by the tangent lines t_{1B}, t_{2B} from the space-like point B to the conic K

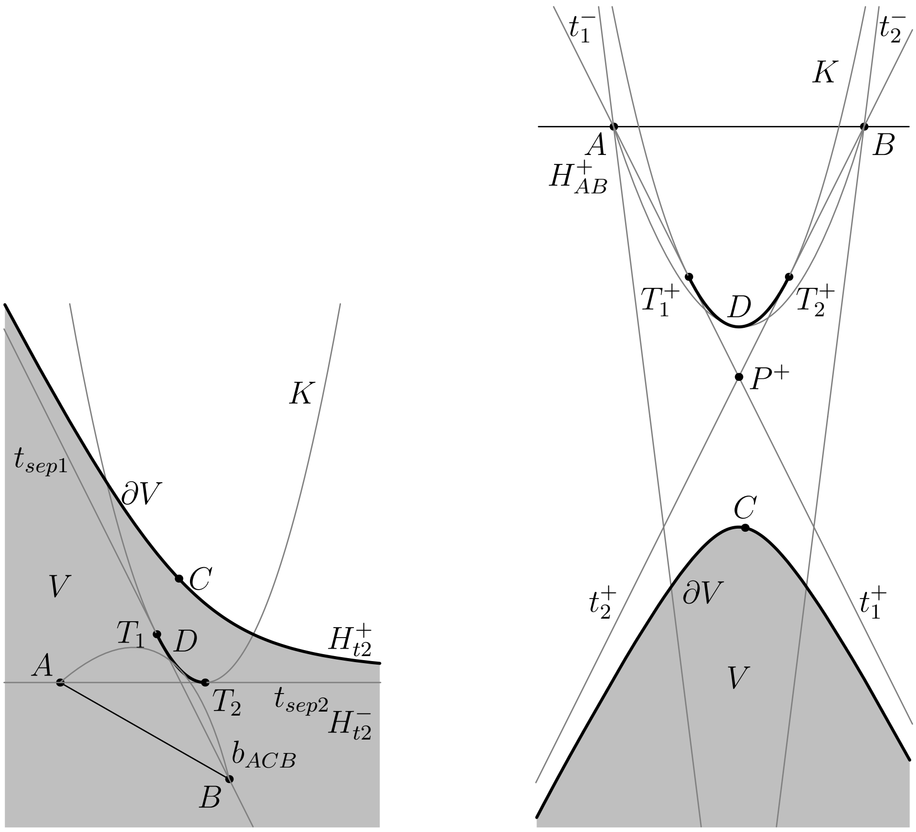
Let us divide the set D_{in} into the interior points of contact which are located in the half-plane $\overline{H^{+}_{AB}}$ (resp. H^{-}_{AB}) denoted by D^{+}_{in} (resp. D^{-}_{in}). There holds $D_{in} = D^{+}_{in} \cup D^{-}_{in}$.

Theorem 3 (Set of interior points of contact)**.** *Let A, B be two different space-like points such that the line $\overleftrightarrow{AB}$ is not tangent to K.*
a) Assume that $AB \cap K = \emptyset$. If non separating tangent lines t_1, t_2 converge, then the set of interior points of contact $D_{in} \neq \emptyset$.
b) Assume that $AB \cap K \neq \emptyset$. If the pair of tangents t^{+}_1, t^{+}_2 converge, then the set $D^{+}_{in} \neq \emptyset$. The same holds for the pair t^{-}_1, t^{-}_2 and the set D^{-}_{in}.

The theorem says when the set of interior points of contact is not empty for A, B such that the line $\overleftrightarrow{AB}$ is not tangent to K. In the case *a)* we consider A lies in one of the areas 1,2,3,4 in the figure 2. , in the case *b)* in one of the areas 5,6.

Hypothesis 1. *In the theorem 3 propositions a), b) hold vice versa for $D_{in} \neq \emptyset$. Then, let t_1, t_2 be a pair of not separate tangents or let the tangents lie in the $\overline{H^{+}_{AB}}$ half-plane or H^{-}_{AB} half-plane. Let t_1, t_2 converge. If there exists a point C such that C and $P = t_1 \cap t_2$ lie in the same half-plane with respect to the $\overleftrightarrow{T_1T_2}$ and b_{ACB} is double contact, then let us denote the relevant points of contact U_1, U_2. The set of interior points of contact (D_{in} or D^{+}_{in} or D^{-}_{in}) is either the continuous arc $\overset{\frown}{T_1T_2}$ on K or the union of the arcs $\overset{\frown}{T_1U_1} \cup \overset{\frown}{U_2T_2}$. The points T_1, T_2 are not in the set of the interior points of contact.*

If D_{in} contains a set $\{\overset{\frown}{T_1U_1} \cup \overset{\frown}{U_2T_2}\}$, we will mark this union of two continuous arcs by *torn arc*. Later, we will see that the union of these two arcs will have the same properties



(a) (b)

Fig. 3. (a) The set $D = \overset{\frown}{T_1T_2}$ consists of one continuous arc of the exterior points of contact
(b) The set $D = \overset{\frown}{T_1T_2}$ consists of one arc of the interior points of contact

as a continuous arc of interior touches $\overset{\frown}{T_1T_2}$. So this torn is just apparent and this union has structural properties for our use as a continuous arc.

Theorem 4 (Set of points of contact)**.** *Let K be the conic section and two different space-like points A, B be such that the line $\overleftrightarrow{AB}$ is not tangent to K. Then, the set of the points of contact D is either one arc of the exterior points of contact or one arc of interior points of contact. The arc of the interior points of contact may be torn.*

Proof. If the point A lies in one of the areas 1, 2, 3, 4 in figure 2. , we have $AB \cap K = \emptyset$. Then, the pair of separate tangents determine a set of the exterior points of contact $D_{out} = \overset{\frown}{T_1T_2}$. The set of interior points of contact $D_{in} = \emptyset$ because of the pair of not separate tangents diverge. So the D consists of one continuous arc of the exterior points of contact, see fig. 3. a.

If $AB \cap K \neq \emptyset$, the point A lies in one of the areas 5, 6 in the figure 2. . Then parabola is unbounded in the one half-plane H_{AB}^-, and bounded in the $\overline{H_{AB}^+}$. Tangents corresponding to half-plane H_{AB}^- always diverge, so there are no interior points of contact. In the half-plane $\overline{H_{AB}^+}$ tagents converge, so the $D_{in} = \overset{\frown}{T_1T_2}$ or its torn version. Using the theorem 2, there are no exterior points of contact. Finally, the set D consists of one arc of interior points of contact which may be torn, see fig. 3. b. $\square$

For the given points $A, B, X \in D \subseteq K$ and the tangent line t at X to K, the Bézier curves touching the conic section K are clearly identified. To find the middle control vertex C, we use the following map σ, called *boundary map*.

Theorem 5. *Let K be conic section and the two different space-like points A, B be such that line $\overleftrightarrow{AB}$ is not tangent to K. Let $X = [x_0, y_0] \in D$. Then the relevant boundary map $\sigma : D \to \mathbb{R}^2_1$ has the form*

$$\sigma(X) = \frac{b(t_0) - B^2_0(t_0)A - B^2_2(t_0)B}{B^2_1(t_0)}\,, \tag{3}$$

where $t_0 \in [0, 1]$ is a solution of the equation

$$0 = \alpha t_0^2 + \beta t_0 + \gamma \tag{4}$$

and

$$\begin{aligned}
\alpha &= 2\,(\alpha_a - \alpha_b)\\
\beta &= 2\begin{pmatrix} x_0 & y_0 & 1 \end{pmatrix}\begin{pmatrix} 2k_A & 2k_B & k_1\\ 2k_B & 2k_C & k_2\\ k_1 & k_2 & k_3 \end{pmatrix}\begin{pmatrix} x_0\\ y_0\\ 1 \end{pmatrix}\\
\gamma &= -\frac{\beta}{2}\,,
\end{aligned}$$

where

$$\begin{aligned}
\alpha_a &= \begin{pmatrix} a_x & a_y & 0 \end{pmatrix}\begin{pmatrix} k_A & k_B & k_D\\ k_B & k_C & k_E\\ k_D & k_E & k_F \end{pmatrix}\begin{pmatrix} x_0\\ y_0\\ 1 \end{pmatrix}\\
\alpha_b &= \begin{pmatrix} b_x & b_y & 0 \end{pmatrix}\begin{pmatrix} k_A & k_B & k_D\\ k_B & k_C & k_E\\ k_D & k_E & k_F \end{pmatrix}\begin{pmatrix} x_0\\ y_0\\ 1 \end{pmatrix}\\
k_1 &= \begin{pmatrix} k_A & k_B & k_D \end{pmatrix}\begin{pmatrix} -a_x\\ -a_y\\ 1 \end{pmatrix}\\
k_2 &= \begin{pmatrix} k_B & k_C & k_E \end{pmatrix}\begin{pmatrix} -a_x\\ -a_y\\ 1 \end{pmatrix}\\
k_3 &= 2\begin{pmatrix} k_D & k_E & k_F \end{pmatrix}\begin{pmatrix} -a_x\\ -a_y\\ 0 \end{pmatrix}.
\end{aligned}$$

The proof can be found in [4]. For every point $X \in K$ there exists at most one point C such that X is the common point of contact of order 2 for the Bézier curve $b_{ACB}(t)$ and K, so the boundary map is well-defined. The range of values of σ consists of one continuous unbounded curve ∂V with degree at most four. The boundary map σ is injective for the points U_i . The curve ∂V divides the plane into two regions W_1, W_2, and one of them is the set of acceptable solution V. The following theorem says which one.

Theorem 6. *Let A, B be two different space-like points, the set D be their set of points of contact and σ generates for them the curve ∂V. Let ∂V divide the plane into two regions W_1, W_2. If $D = D_{out}$, then $W_i \subseteq V$ if $A, B \in W_i$. If $D = D_{in}$, then $W_i \subseteq V$ if $A, B \notin W_i$.*

The proof can be found in [4]. Finally, for the two space-like points A, B and the conic K the set of acceptable solutions V consists of one region.

4.1. Special cases

In the text above, we considered the point A lying in one of the areas 1,...,6 in the figure 2. . We left out the case $\overleftrightarrow{AB}$ to be tangent to K, i.e. $\overleftrightarrow{AB} \cap K = \{T\}$. We can not map T by σ, because there are many points C (they form a half-line on $\overleftrightarrow{AB}$) such that T is the common point of contact of order 2 for the Bézier curve $b_{ACB}(t)$ and K. In this case, the arc D is determined by tangent line through T. This line replaces in above situations (in theorems 2, 3) either one of the converging or one of the separating lines. Then we have to change D by $D \setminus \{T\}$ in the theorem 6. So the curve ∂V contains a part of the line $\overleftrightarrow{AB}$ for the case $T \notin AB$, for the case $T \in AB$ is $\partial V = \overleftrightarrow{AB}$. If $A = B$ the ∂V contains the parts of the tangents from this point to K.

5. Conclusion

We described an area of all such points C that a quadratic Bézier curve with control points A, C, B is space-like on condition the plane generated by A, C, B cuts the light cone Q in a parabola. This work covers one of the cases omitted in work [4]. The characterization is described via method of contact point and all the results mentioned above are proven either this paper or in [4].

When the case of hyperbolic intersection of the light cone and the plane ρ will be solved, then the case of quadratic space-like curves will be covered.

References

[1] BERGER, M. *Geometry I., II.* Berlin: Springer-Verlag, Universitext, 1987. Translated from the French by M. Cole and S. Levy.

[2] DUBROVIN, B. A., FOMENKO, A. T., NOVIKOV, S. P. *Modern Geometry – Methods and Applications: Part I: The Geometry of Surfaces, Transformation Groups, and Fields* New York: Springer-Verlag, second edition, 1992. Translated from the Russian by Robert G. Burns.

[3] GALLUSOVÁ, B. Bézierove krivky a ich vlastnosti v Minkowského priestore In *Zborník príspevkov*, Študentská vedecká konferencia FMFI UK, Bratislava 2010.

[4] POKORNÁ, B. *Quadratic Bézier curves in the three dimensional Minkowski space* Rigorózna práca, 2011. In preparation.