PROCEEDINGS OF SYMPOSIUM ON COMPUTER GEOMETRY SCG' 2012, VOLUME 21
pp. 93 - 98

PLANAR CUBIC SPACE-LIKE BÉZIER CURVES IN THREE DIMENSIONAL MINKOWSKI SPACE

Barbora Pokorná [1], **Pavel Chalmovianský** [2]

KAGDM, FMFI UK, Mlynská dolina, 842 48 Bratislava, SR

[1] e-mail: barbora.pokorna@fmph.uniba.sk

[2] e-mail: pavel.chalmoviansky@fmph.uniba.sk

Abstract. This paper considers planar cubic Bézier curves in three-dimensional Minkowski space. We shall consider the conditions for the control points A, C, D, B of the Bézier curve such that the Bézier segment is pointwise space-like. For the control point C, we shall give a geometrical interpretation of the feasibility condition provided the rest of the control points are fixed.

Keywords: cubic Bézier curve, space-like curve, conic section.

1. Introduction

We will continue in the spirit of the works [3, 6] where the set of admissible points C is described for quadratic space-like Bézier curves. In this paper, we show the set of admissible points for planar cubic space-like Bézier curves. At first, we recall some facts about Minkowski space, more on this topic can be found in the books [1, 2]. Then, we will show, which methods and propositions used in the quadratic case can be used in cubic case. At the end, we give a few specific examples.

2. Minkowski space, Bézier curves and their properties

Pseudo-Euclidean space, denoted by $\mathbb{R}^n_p$, $n \in \mathbb{N} = \{1, 2, 3, \dots\}$, $p \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ is an $n-$dimensional real vector space with a regular quadratic form $\mathbf{q}\colon \mathbb{R}^n \to \mathbb{R}$, where $\mathbf{q}(x_1, \dots, x_n) = \sum_{i=1}^{n-p} x_i^2 - \sum_{j=n-p+1}^{n} x_j^2$ in certain basis. For $p = 1$, it is called *Minkowski space*. For $p = 0$, we get *Euclidean space*. We use the notation $\overline{x} = (x_1, \dots, x_n)^\top$ for the vectors in $\mathbb{R}^n_p$.

Let $M_{n,n}(\mathbb{R})$ be the set of $n \times n$ matrices with real coefficients. We can write the quadratic form $\mathbf{q}$ in a certain basis of $\mathbb{R}^n$ in the matrix form as $\mathbf{q}(\overline{x}) = \overline{x}^\top M \overline{x}$, where $M \in M_{n,n}(\mathbb{R})$ is symmetric and regular. A quadratic form has an associated polar form $P\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by $P(\overline{x}, \overline{y}) = \overline{x}^\top M \overline{y}$. Clearly, it is bilinear and symmetric. Since $\mathbf{q}(\overline{x}) = P(\overline{x}, \overline{x})$, the polar form plays a role of scalar product. Hence, we call it *pseudo-scalar product* since positive definiteness might not be satisfied. The vectors $\overline{x}, \overline{y} \in \mathbb{R}^n_p$ are *pseudo-orthogonal* if $P(\overline{x}, \overline{y}) = 0$.

By standard construction, we get an affine space with a *pseudo-Cartesian* coordinate system $S(O, x_1, \dots, x_n)$, where the directions of axes $x_1, \dots, x_n$ are pseudo-orthogonal and

[1] This project has been supported by the project UK/328/2012.

[2] This research was partly funded by the project ASFEU Comeniana - methods and means of presentation and digitization of 3D cultural heritage objects, OPVaV-2011/4.2/07-SORO, ITMS: 26240220077.

O is the origin. We say, that the coordinate axes $x_1, \ldots, x_{n-p}$ are space-like and the axes $x_{n-p+1}, \ldots, x_n$ are time-like. In the following, we work using such a setup.

Using the quadratic form, we classify the vectors in the pseudo-Euclidean space. We call the vector $\overline{x} \in \mathbb{R}^n_p$ *space-like* if $\mathbf{q}(\overline{x}) > 0$, *time-like* if $\mathbf{q}(\overline{x}) < 0$ and *light-like* if $\mathbf{q}(\overline{x}) = 0$. All the vectors in $\mathbf{q}^{-1}(0)$ are also called *isotropic*. The set of all light-like vectors forms an *isotropic cone* Q of the corresponding quadratic form $\mathbf{q}$.

A point $x \in \mathbb{R}^n_p$ is space-like (time-like, light-like respectively) if its position vector $\overline{x} = x - O$ is such. Note that this depends on the coordinate system. There are two possible ways, how to define space-like curve (time-like and light-like respectively). A differentiable curve $p\colon I \to \mathbb{R}^n_p$ is called *tangentially space-like* if the tangent vector $\dot{p}(t)$ is space-like for each $t \in I$. A curve $p\colon I \to \mathbb{R}^n_p$ is called *pointwise space-like* if it contains only space-like points, i.e. the vector $\overline{p(t)} = p(t) - O$ is space-like vector for every $t \in I$. In our work, we use the definition of the pointwise space-like curve. One of the advantages is that the condition of differentiability is not required, although the curves we consider in this paper are polynomial.

Bézier curve in Minkowski space of degree n is the polynomial map $b : [0, 1] \to \mathbb{R}^n_1$ that $b(t) = \sum_{i=0}^{n} B^n_i(t)\, b_i$ for $t \in [0, 1]$. Points $b_i \in \mathbb{R}^n_1$ are called control points, $B^n_i(t) = \binom{n}{i}(1-t)^{n-i}t^i$ for $i \in \{0, \ldots, n\}$ are Bernstein polynomials of degree n. The Bézier curve $b(t)$ always passes through the first and the last control point and it lies within the convex hull of its control points. The construction of Bézier curves is invariant under affine transformations, but not invariant under all projective transformations. More properties can be found in [4, 5].

There is an alternative way how to define the Bézier curve. Instead of one control point, we determine one point of the curve and the tangent line to the curve at this point. We focus on the planar cubic curves. Let the points $A, F, B, T \in \mathbb{R}^2_1$ and ℓ_T be a line such that the point $T \in \ell_T$. The question is whether there is an appropriate control point C, such that the Bézier curve $b_{ACFB}(t)$ exists and how many such points C exist. It depends only on the direction vector of the line ℓ_T.

Definition 2.1 (Feasible tangent line at a point of the cubic)**.** Let the points $A, F, B, T \in \mathbb{R}^2_1$ and ℓ_T be a line such that the point $T \in \ell_T$. The line ℓ_T is called *feasible*, if there exist at least one point C such that there exist the Bézier curve $b_{ACFB}(t)$ containing the point T with tangent line ℓ_T at the point T.

There may exist more control points C for the given points $A, F, B, T \in \mathbb{R}^2_1$ and a feasible tangent line ℓ_T, see fig. 1(a).

3. Space-like conditions

Let us consider Minkowski space $\mathbb{R}^3_1$ and a cubic Bézier curve with the control points A, C, F, B in this order, denoted by $b_{ACFB}(t)$. In order a Bézier curve to be space-like, each of its points have to be space-like. Let the points $A = [a_1, a_2, a_3]$ and $B = [b_1, b_2, b_3]$ be fixed. Since Bézier curve interpolates its endpoints, we have necessary and sufficient conditions

$$a_1^2 + a_2^2 - a_3^2 > 0\,, \tag{1}$$

$$b_1^2 + b_2^2 - b_3^2 > 0\,. \tag{2}$$

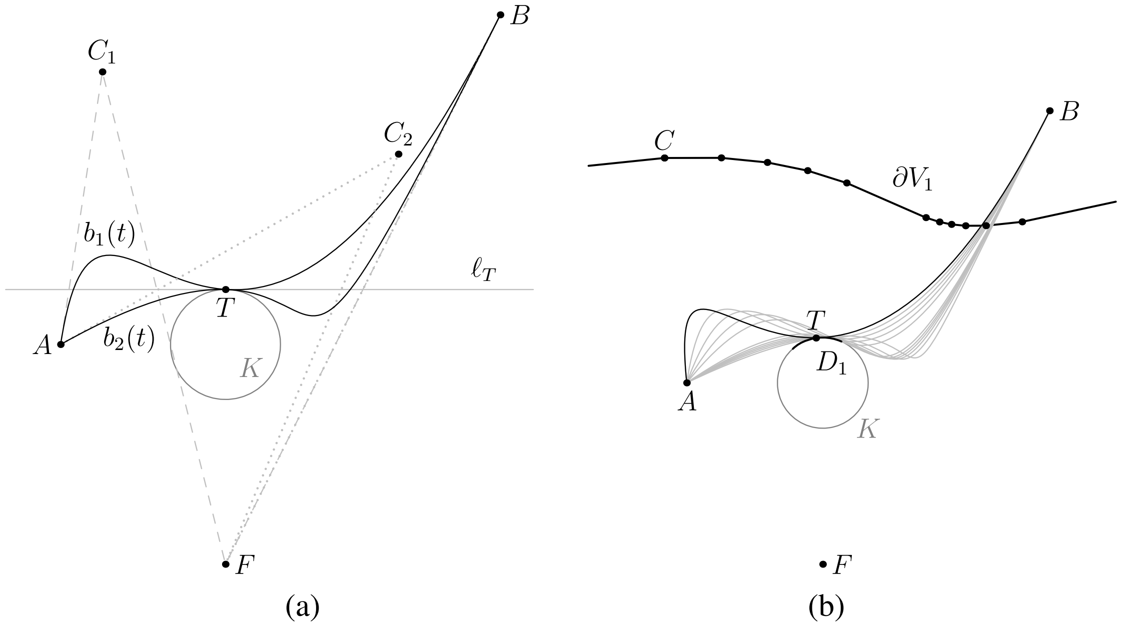

Fig. 1. (a) For *feasible* tangent line ℓ_T, there may exist two (or more) points C_1, C_2 such that the Bézier curves $b_1(t), b_2(t)$ contain the point T and they have tangent line ℓ_T at the point T.
(b) The boundary map σ maps the part of the set of points of contact $D_1 \subset D$ on the curve $\partial V_1 \subset \partial V$. See that $\sigma(T) = C$.

At the beginning, we consider only the case that Bézier curve is planar and it lies in the affine plane $\rho \subset \mathbb{R}^3_1$. We fix the points A, F, B.
In any case, the intersection of the light-cone Q and the plane ρ is a conic section K (see fig. 2(a)). The figure 2(b) shows all cases how the set of all space-like points S in the possible types of the plane ρ looks like. The pointwise space-like Bézier curve $b_{ACFB}(t) \subset S$. We solve the problem in the plane ρ for each type of conic section and the planar results can be put together to form the spatial result.

Let $S_\rho(O, x, y)$ be any pseudo-Cartesian coordinate system in the plane ρ. Let $A = [a_x, a_y]$, $C = [c_x, c_y]$, $F = [f_x, f_y]$ a $B = [b_x, b_y]$ be the local affine coordinates of the control points in $S_\rho(O, x, y)$. From now on, the points A, B, F are arbitrary, but fixed, and they satisfy the conditions (1), (2).

Definition 3.1 (Set of admissible solutions)**.** Let $V_\rho(A, F, B)$ be a set of points $C \in \rho$ such that the curve b_{ACFB} is space-like. Then, we say that $V_\rho(A, F, B)$ is a *set of admissible solutions* in the plane ρ with respect to A, F, B. If no confusion arise, we say only the set of admissible solutions V.

Definition 3.2. By $V^v_\rho(A, F, B)$, we denote the set of points $C \in \rho$ such that $b_{ACFB} \cap K = M$, where $X \in M$ is a point of contact of order 2 (or higher) between $b_{ACFB}(t)$ and K. We denote the set of points $C \in \rho$ such that b_{ACFB} and K have transversal intersection by $V^t_\rho(A, F, B)$.

Note 3.1. The set M in the definition 3.2 contains at most three points, since two different curves (quadratic and cubic) may have at most three common points of contact of order 2 (see

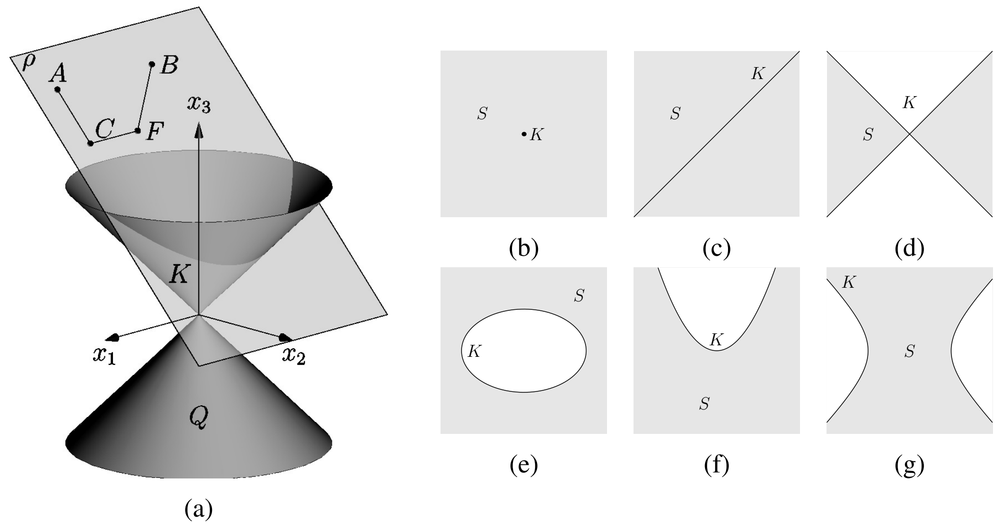

Fig. 2. (a) Plane ρ spans the points A, C, F, B. In the case of their non-collinearity, they generate ρ as their affine hull. The conic section K is an intersection of the light cone Q and the plane ρ.
(b–g) Let $K \subset \rho$ be the conic section (point, double line, pair of lines, ellipse, parabola, hyperbola). The set S consists of all space-like points in the plane ρ.

e.g. Bézout theorem in [5]). If we mark a point by the letter T, we mean a point of contact from M.

The union of disjoint sets $V_\rho(A, F, B) \cup V_\rho^v(A, F, B) \cup V_\rho^t(A, F, B)$ gives the whole plane ρ for the given points A, F, B.

4. Set of admissible control points C

We study the set $V_\rho^v(A, F, B)$. It is natural, because a "boundary" between the situation that two curves have no common points and the situation that one curve intersects the other curve is, that they touch each other in our setup.

Definition 4.1 (Set of points of contact)**.** We say that the set $D \subset K$ is the *set of points of contact* between K and the set of all b_{ACFB} if for any point $X_0 = [x_0, y_0] \in D$ there is at least one point C_0 such that $C_0 \in V_\rho^v(A, F, B)$ and $X_0 \in b_{AC_0FB} \cap K$.

For now, we are unable to describe the exact shape of the set D. But we create an experimental program, which enables users an interactive work with the control points of the Bézier curve and conic sections. First, the user determine the position of the control points A, F, B and the type of conic section. Then, the user changes the position of the control point C by moving the mouse and the Bézier curve $b_{ACFB}(t)$ is drawn immediately. In this way, it is possible to observe the shapes of the set D and the curves formed by the corresponding points C. The example of the set D obtained in this way can be seen in fig. 3.

When we find the set D, we need to obtain the corresponding points C forming the boundary of the set $V_\rho(A, F, B)$. For the given points A, F, B, $X \in D \subseteq K$ is the tangent line ℓ_X at X to K *feasible*. In order to find the control point C, we use the following map σ.

Definition 4.2 (Boundary map). Let D be the set of points of contact for the given points A, F, B and K and let $\mathcal{P}(\rho)$ be the power set of the plane ρ. The map $\sigma : D \to \mathcal{P}(\rho)$ is called *boundary map* if for every $X \in D$ holds $\sigma(X) = \{C \in \rho \mid C$ satisfies the definition $4.1\}$, see fig. 1(b).

Theorem 4.1. Let the point $X = [x_0, y_0] \in D$. Then, the corresponding boundary map $\sigma : D \to \mathcal{P}(\rho)$ has the form

$$\sigma(X) = \{\frac{b(t_0) - B_0^3(t_0)A - B_2^3(t_0)F - B_3^3(t_0)B}{B_1^3(t_0)}, t_0 \in (0,1)\}, \tag{3}$$

such that t_0 is a solution of the equation

$$0 = \alpha t_0^3 + \beta t_0^2 + \gamma t_0 + \delta \tag{4}$$

and for $A = [a_x, a_y, 1],\ F = [f_x, f_y, 1],\ B = [b_x, b_y, 1],\ X = [x_0, y_0, 1]$ are

$$\begin{aligned}
\alpha &= (A - 3F + 2B)M_K X^\top, \\
\beta &= 3(F - A)M_K X^\top, \\
\gamma &= 3(A - X_0)M_K X^\top, \\
\delta &= -\frac{\gamma}{3}.
\end{aligned}$$

Proof. Since we consider only affine points $C \in \rho$, let

$$K = \{[x, y] \in \mathbb{R}^2 : k_A x^2 + 2k_B xy + k_C y^2 + 2k_D x + 2k_E y + k_F = 0\}$$

with matrix M_K. Because of the point of the contact $X \in b_{ACFB}(t)$, there exists $t_0 \in (0, 1)$ such that $X = b_{ACFB}(t_0) = B_0^3(t_0)A + B_1^3(t_0)C + B_2^3(t_0)F + B_3^3(t_0)B$. The point $X \in K$ is light-like so $X \notin \{A, B\}$ and $t_0 \notin \{0, 1\}$. The equality $\left\langle \nabla f(x_0, y_0), \frac{d}{dt} b_{ACFB}(t_0) \right\rangle = 0$ holds, because $X = [x_0, y_0] \in D$. From this cubic equation, we obtain three roots $t_0^0, t_0^1, t_0^2 \in \mathbb{C}$. The line ℓ_X is feasible, so at least one (at most three) $t_0^i \in (0, 1)$, $i \in \{0, 1, 2\}$. For each $t_0^i \in (0, 1)$ we obtain the relevant point C_i using the equation (3). Each point C_i satisfies the definition 4.1 for the point of the contact X. Hence, $\sigma(X) = \{C_i \mid t_0^i \in (0, 1)\} \neq \emptyset$. □

At the end, we show one example (see fig. 3). The points $A = [-3, 0],\ F = [0, -4]$, $B = [5, 6]$. The approximation of the set $D = \{X = [x, y] \in K \mid x \in \langle -0.66, 0.41\rangle$ for $y > 0$ and $x \in \langle -0.4, 0.84\rangle$ for $y < 0\}$ was founded using experimental program and verify using *Asymptote* with an accuracy ± 0.02. Using the boundary map σ, we compute the points $C \in V_\rho^v(A, F, B)$. In the future we might prove, they form the boundary ∂V of the set of admissible solutions V.

5. Conclusion

We explore how the set of admissible solutions $V_\rho(A, F, B)$ looks like in the experimental program. Based on observations and previous work with quadratic curves, we start to study the set $V_\rho^v(A, F, B)$. We define the necessary conditions for the control points A, B and the equations of the boundary map σ.

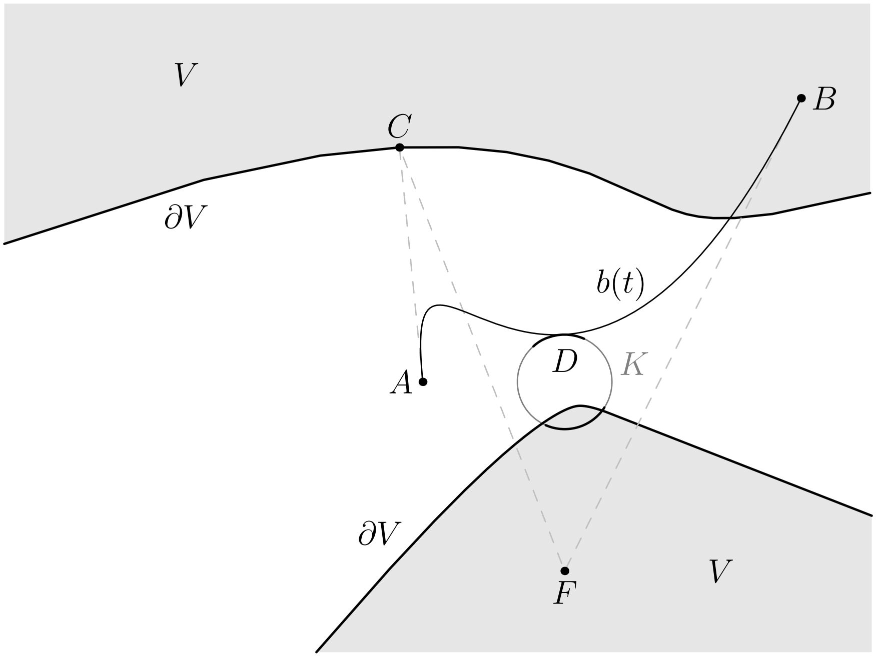

Fig. 3. The set of points of contact D was approximated experimentally, it consists of two arcs. Using the boundary map σ the set D generates two curves forming the boundary $\partial V_\rho(A, F, B)$. The set of admissible solutions $V_\rho(A, F, B)$ consists of two regions.

In the future, we need to determine the conditions for the direction vector of the line ℓ_T, in order to be feasible for given points A, F, B, T. Then we identify the exact shape of the set D by choosing those points $T_0 \in K$ that the corresponding tangent line ℓ_{T_0} to the K at the point T_0 is feasible for the points A, F, B, T_0. At the end, we prove that the set $V_\rho^v(A, F, B)$ obtaining as $\sigma(D)$ is the boundary of the set of admissible solutions $V_\rho(A, F, B)$.

References

[1] BERGER, M. *Geometry. I., II.* Universitext, Berlin: Springer-Verlag, 1987, translated from the French by M. Cole and S. Levy.

[2] CHALMOVIANSKÝ, P. Pseudo-euclidean spaces and hyperbolic geometry. *Proceedings of Symposium on Computer Geometry 19*, 2010.

[3] CHALMOVIANSKÝ, P., POKORNÁ, B. Quadratic space-like Bézier curves in the three dimensional Minkowski space. *Proceedings of Symposium on Computer Geometry 20*, 2011.

[4] FARIN, G., HOSCHEK, J., KIM, M.-S. *Handbook of computer aided geometric design*. Amsterdam: North-Holland, 2002.

[5] KUNZ, E. *Introduction to plane algebraic curves*. Boston, MA: Birkhäuser Boston Inc., 2005, translated from the 1991 German edition by Richard G. Belshoff.

[6] POKORNÁ, B. Quadratic space-like Bézier curves in three dimensional Minkowski space, 2012, project of Dissertation Thesis.