# Avoiding Quadratic Obstacles in the Euclidean Plane Using Cubic Bézier Paths

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**Abstract**. Cubic Bézier curves as collision-free paths are widely used in path planning. The essential task for finding all possible collision-free paths is necessary to find those paths, which only touch an obstacle. We solve the planar cases for an obstacle represented by conic section $K$ as bounding object. The cubic path is represented by a Bézier curve with control points $A, C, F, B$, where $A, B$ are given start and goal positions and the point $F$ is arbitrary, but fixed. This paper describe the set $D$ of points at conic section $K$, which are admissible points of contact, and the corresponding point $C$ for $X \in D$.

*Keywords*: cubic Bézier curve, collision-free path, conic section.

## 1 Introduction

Motion planning is a fundamental research area in robotics. A motion plan involves determining what motions are appropriate for the robot so that it reaches a goal state without colliding into obstacles [5]. Let $\mathbb{R}^2$ be the Euclidean plane with obstacle represented by a conic section $K$ as bounding object. Let the point $A$ be the start and the point $B$ be the finish. We find all cubic Bézier paths starting at $A$ and ending at $B$ representing collision-free path with respect to an obstacle $K$.

## 2 Notation and problem definition

Let $\mathbb{R}^2$ be an affine Euclidean plane formed by points $X = [x, y]$. Let $\boldsymbol{Q_K} \in M_{3,3}(\mathbb{R})$ be a symmetric matrix. The algebraic curve of degree 2 called *conic section* is the set $K = \{[x, y] \in \mathbb{R}^2 \colon f(x, y) = 0 \text{ for } f(x, y) = (x\, y\, 1)\boldsymbol{Q_K}(x\, y\, 1)^\top\}$. More about spaces with quadratic form can be found in [1]. In appropriate cases, we consider the equation of the conic section instead of $K$ due to the fact that the field $\mathbb{R}$ is not algebraically closed. The conic section is the set of self-polar points with respect to polar form $P(X, Y)$ determined by the matrix $\boldsymbol{Q_K}$. We say that the point $X$ lies out of conic section if $P(X, X) > 0$. We denote $P_A = P(A, X) = A\boldsymbol{Q_K}X^\top$, when $X = (x, y, 1) \in K$ and $A = (a_x, a_y, 1)$. For the point $Y$, the $Y^\perp$ is the polar line determined by equation $(Y, 1)\boldsymbol{Q_K}(X, 1)^\top = 0$. More about conic sections can be found in [2, 4].

*Bézier curve* of degree $n$ in the space $\mathbb{R}^d, d \in \mathbb{N}, d \geq 2$ is a polynomial map $b \colon [0, 1] \to \mathbb{R}^d$ given by $b(t) = \sum_{i=0}^{n} B_i^n(t)\, V_i$. The points $V_i \in \mathbb{R}^d$

for $i \in \{0, \ldots, n\}$ are called *control points*, the functions $B_i^n(t) = \binom{n}{i}(1-t)^{n-i}t^i$ are Bernstein polynomials of degree $n$. More about properties of Bézier curves can be found in [6].

## 3 Cubic collision-free path

Let $\rho$ be a Euclidean plane containing the conic section $K$. Let the points $A, B \in \rho$ lie out of conic section. Let the point $F$ be arbitrary, but fixed. We need to find the *set of admissible solutions* $V_\rho(A,F,B)$ of such points $C \in \rho$, that the curve $b_{ACFB}$ is collision-free with respect to $K$. We start by searching the boundary of this set.

By $V_\rho^v(A,F,B)$, we denote the set of points $C \in \rho$ such that $b_{ACFB} \cap K$ contains only the points of contact of order 2 between the Bézier curve and the conic section. We say that the set $D \subset K$ is the *set of points of contact* between $K$ and the set of all $b_{ACFB}$ if for any point $X \in D$, there is a point $C$ such that $C \in V_\rho^v(A,F,B)$ and $X \in b_{ACB} \cap K$. The exact shape of the set $D$ is shown later.

At first, we find the map $\sigma\colon D \to V_\rho^v(A,F,B)$, which express the correspondence between the points of contact with $K$ and the middle control points. The boundary of the set $V_\rho(A,F,B)$ is $\partial V = V_\rho^v(A,F,B)$.

**Definition.** Let $D$ be the set of points of contact for the given points $A, F, B$ and $K$ and let $\mathcal{P}(\rho)$ be the power set of the plane $\rho$. The map $\sigma : D \to \mathcal{P}(\rho)$ is called *boundary map* if for every $X \in D$ holds $\sigma(X) = \{C \in \rho \mid C \in V_\rho^v(A,F,B)$ and $X \in b_{ACFB} \cap K$ is the point of contact$\}$.

**Theorem.** Let the point $X \in D \subset K$ whereas the conic section $K$ be represented with matrix $\boldsymbol{Q_K}$. Let the real numbers

$$\begin{aligned}
\alpha &= (A - 3F + 2B)\boldsymbol{Q_K}X^\top, \\
\beta &= (F - A)\boldsymbol{Q_K}X^\top, \\
\gamma &= A\boldsymbol{Q_K}X^\top, \\
\delta &= -\gamma
\end{aligned}$$

be the coefficients of the cubic function

$$R(t) = \alpha t^3 + 3\beta t^2 + 3\gamma t + \delta \tag{1}$$

for $A = [a_x, a_y, 1]$, $F = [f_x, f_y, 1]$, $B = [b_x, b_y, 1]$, $X = [x_0, y_0, 1]$. Then, the corresponding boundary map $\sigma : D \to \mathcal{P}(\rho)$ has the form

$$\sigma(X) = \left\{ \frac{b(t_0) - B_0^3(t_0)A - B_2^3(t_0)F - B_3^3(t_0)B}{B_1^3(t_0)},\ t_0 \in (0,1) \wedge R(t_0) = 0 \right\}. \tag{2}$$

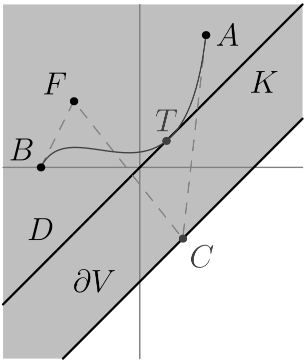


Figure 1: The boundary $\partial V$ of the set of admissible solutions is a parallel line to the double line $p$ for $K = p$.

We can write the discriminant of the cubic equation $R(t)$ as $\Delta = 108P(A,X)(P^3(F,X) - P(A,X)P^2(B,X))$ using the notation of polar lines equation. This discriminant enables to compute the number of real roots of the function given by (1) over an interval. Combining with Budan-Fourier theorem [3] applied on interval $\langle 0,1\rangle$, we are able to determine the number of roots lying in $(0,1)$. In other words, we know how many points $C_i$ exist for given $X \in K$.

### 3.1 Singular conic sections

At first, we find the set $D$ for singular conic sections, then we consider regular conic sections.

**Theorem.** If the conic section $K = p$, the set of admissible points of the contact $D = K$. Moreover, the boundary of the set of admissible solutions $\partial V$ is a parallel line to the double line $p$ (see fig. 1).

**Proof.** Without loss of generality, let us consider the conic section $K\colon -x + y = 0$. We obtain $P(A,X) = \frac{1}{2}(-a_x + a_y)$, which is the constant independent on the choice of $X$. Similarly, the expressions $P(B,X)$ and $P(F,X)$ are constants. Hence, the coefficients $\alpha, \beta, \gamma, \delta$ in (2) are constants independent on the point $X \in D$. Hence, the solutions $t_1, t_2, t_3$ of the equation (1) are constants for all $X \in D \subset K$.

Now, we need to prove that $D = p$ and for every $X \in D$ exists exactly one $i \in 1, 2, 3$ such that root $t_i \in (0,1)$. Let us count the number of roots of the equation (1) belonging to $(0,1)$. Computing the values derivatives of $R(t)$ at the end points of the interval $\langle 0,1\rangle$, we obtain the table 1. In the case of singular conic sections all the values are constants independent

| | $t=0$ | $t=1$ |
|---|---|---|
| $R(t)$ | $-P_A$ | $2P_B$ |
| $R'(t)$ | $3P_A$ | $-3P_F+6P_B$ |
| $R''(t)$ | $-6P_A+6P_F$ | $-12P_F+12P_B$ |
| $R'''(t)$ | $6P_A-18P_F+12P_B$ | |
| $\Delta$ | $108P_A(P_F^3-P_AP_B^2)$ | |

Table 1: The values of derivatives of the function $R(t)$ at the end points of the interval $\langle 0,1\rangle$ and the value of the discriminant.

on the point $X\in D$.

The assumption of the Budan-Fourier theorem reads that the product $R(0)R(1)\neq 0$. It holds iff $P_A\neq 0\wedge P_B\neq 0$. This is accomplished, because the points $A,B\notin K$. Now, we consider some configurations of the points $A,F,B$ with respect to $K$ and check the corresponding number of roots of the equation (1) in the interval $\langle 0,1\rangle$. For the obtaining of the collision-free path, the points $A,B$ must lie in the same half plane with respect to $K$, so we assume $P_AP_B>0$.

For $P_F=0$, the number of sign changes is equal to 3 and the discriminant $\Delta<0$. Which means, there is exactly one real root $t_0$ within the interval $\langle 0,1\rangle$ and the uniquely defined Bézier curve always exists. Let $P_F\neq 0$ and without loss of generality let $P_F>0$. If $0<P_A<P_F$, we distinguish these two possible positions of the point $B$ as $0<P_F<P_B$ and $0<P_B<P_F$. The corresponding table shows that there is exactly one real root $t_0$. If $0<P_F<P_A$, we distinguish two possible positions of the point $B$ the same way. The number of sign changes is either 3 or 1, but in the case of 3 the discriminant $\Delta<0$. It restricts the number of roots to 1. If $P_A<0<P_F$, the point $B$ must be in the same half-plane, so $P_B<0$ and there is only one real root within $\langle 0,1\rangle$. The conclusion of all the cases is, that the set of points of contact $D=p$ and for every $X\in D$ exists exactly one Bézier curve $b_{ACFB}$, where $C=\sigma(X)$.

At the end, we determine the shape of the curve $\partial V$. Let $T_0\in D$ be an arbitrary fixed point of contact and let $C_0$ be the corresponding middle control point. Let $T\in D$ be arbitrary point of contact different from $T_0$. We express $T=T_0+u\boldsymbol{s_p}$, where $0\neq u\in\mathbb{R}$ and $\boldsymbol{s_p}$ is direction vector of the line $p$. The corresponding point $C$ is obtained from formula (2) and for $t_0\in(0,1)$ is $B_1^3(t_0)>0$. If we substitute $T$ by $T_0+u\boldsymbol{s_p}$ and $T_0$ by $B_0^3(t_0)A+B_1^3(t_0)C_0+B_2^3(t_0)F+B_3^3(t_0)B$, we obtain $C=C_0+\frac{u}{B_1^3(t_0)}\boldsymbol{s_p}$. Hence, the boundary of the set of admissible solutions $\partial V$ is the line with the same direction vector as the line $p$.

**Theorem.** Let $K=p\cup r$. The set of points of contact $D=\overrightarrow{S_pP}\cup\overrightarrow{S_rR}$

| | $P_F < P_A$<br>$P_F < P_B$ | | $P_F < P_A$<br>$\frac{1}{2}P_F < P_B < P_F$ | | $P_F < P_A$<br>$P_B < \frac{1}{2}P_F$ | |
|---|---|---|---|---|---|---|
| | $t = 0$ | $t = 1$ | $t = 0$ | $t = 1$ | $t = 0$ | $t = 1$ |
| $R(t)$ | $-$ | $+$ | $-$ | $+$ | $-$ | $+$ |
| $R'(t)$ | $+$ | $+$ | $+$ | $+$ | $+$ | $-$ |
| $R''(t)$ | $-$ | $+$ | $-$ | $-$ | $-$ | $-$ |
| $R'''(t)$ | $+$ | $+$ | $+$<br>$-$ | $+$<br>$-$ | $+$<br>$-$ | $+$<br>$-$ |
| # of sign changes | 3 | 0 | 3<br>2 | 2<br>1 | 3<br>2 | 2<br>1 |
| sign of $\Delta$ | $-$ | | no influence | | no influence | |

Table 2: The sign changes and the sign of discriminant for the arc of regular $K$ such that $P_A > 0$, $P_B > 0$ and $P_F < P_A$.

(in special case $S_p = S_r = p \cap r$). From the previous lemma, the set $\partial V$ consists of two half-lines parallel with $p$, resp. $r$, connected in the point $C_u$. If the conic section $K = \{[0,0]\}$, then the set $D = \{[0,0]\}$ and the boundary of the set of admissible solutions $\partial V$ is one continuous curve.

### 3.2 Regular conic sections

Now, we need to find the set $D \subset K$ for regular conic sections. Let us focus on the necessary algebraic conditions for $X \in K$ to be $X \in D$. In the case of regular conic sections, the coefficients $\alpha, \beta, \gamma, \delta$ of the function $R(t)$ are linear functions of $X$ in generic case, because the polar forms $P_A, P_B, P_F$ depend on the choice of $X \in D$.

Similarly, we use the table 1 for determination of sign changes of derivatives of the function $R(t)$ in the end points of the interval $\langle 0, 1\rangle$. We must distinguish several cases with respect to the mutual position of the point $X \in K$ and the polar lines $A^\perp, B^\perp$. The polar lines $A^\perp, B^\perp$ divide the conic section $K$ into several arcs and we need to find suitable candidates for the set $D$ between them or their subset. The table 2 shows the sign changes and the determinant for the arc such that $P_A > 0$, $P_B > 0$ and $P_F < P_A$. We create similar tables for each arc of the conic section. We look for all the arcs, where at least one real solution exists within $\langle 0, 1\rangle$. Based on these tables, we can formulate the next theorem.

**Theorem.** The set of admissible points of the contact $D$ is the subset of the union of the arcs $K_i \subset K$ for $i = 1, \ldots, 4$, where

$$K_1 = \{X \in K\colon P_A \geq 0 \wedge P_B \geq 0\},$$
$$K_2 = \{X \in K\colon P_A \leq 0 \wedge P_B \leq 0\},$$
$$K_3 = \{X \in K\colon P_A \geq 0 \wedge P_B \leq 0 \wedge P_F^3 - P_A P_B^2 \geq 0\},$$
$$K_4 = \{X \in K\colon P_A \leq 0 \wedge P_B \geq 0 \wedge P_F^3 - P_A P_B^2 \leq 0\}. \quad \square$$

The equalities in above theorem may occur, because the end points of the arcs may belong to the set $D$. Depending on the positions of the points $A, B, F$, some of these sets may be empty. For example, the set $K_1$ is empty iff the segment $AB$ is a secant of $K$. The necessary condition for the set $D$ is not the sufficient condition simultaneously. It may happened, that the Bézier curve determined by the point $X \in \bigcup_{i=1}^{4} K_i$ has some transversal intersection with $K$. The sufficient conditions are required, because we need to know the exact shape of the set $D$ for computing the boundary $\partial V(A, F, B)$. We plan to find them in the further research.

## 4 Conclusion

We focused on collision-free path finding with respect to quadratic obstacles using cubic Bézier curves. We looked for the set $V(A, F, B)$ containing the admissible middle control points $C$ of collision-free paths. We determined this set for singular conic sections as obstacles. We defined the boundary map $\sigma$ and the necessary conditions for the set of admissible points of contact $D$ for regular conic sections, while $\partial V(A, F, B) = \sigma(D)$. The finding of sufficient conditions for the set $D$ is the topic for further research.

## Acknowledgements

The first author has been supported by the project KEGA 094UK–4/2013 "Ematik+, Continuing education of mathematics teachers". The second author has been supported by the project VEGA 01/0330/13.

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