COMENIUS UNIVERSITY IN BRATISLAVA

Faculty of Mathematics, Physics and Informatics

Quadratic space-like Bézier curves in three dimensional Minkowski space

Project of Dissertation Thesis

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Bratislava 2012

Declaration

I hereby declare I wrote this Project of Dissertation Thesis by myself, only with the help of referenced literature, under the supervision of my supervisor.

Bratislava, 27/02/2012 ....

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Acknowledgements

I would like to thank to my supervisor doc. RNDr. Pavel Chalmovianský, PhD. for his great help, advices and supervising.

Abstract

This paper considers quadratic Bézier curves in three-dimensional Minkowski space. We shall show the conditions for the control points A, C, B of the Bézier curve such that the Bézier segment is space-like. For the middle control point C, we shall give a geometrical interpretation of the feasibility condition. The work continues in the spirit of the works [Gal10], [Pok11], [CP11]. We propose extensions of the topics and methods used in the work for other classes of polynomial or rational curves.

Key words: *Bézier curve, Minkowski space, space-like curve*

Contents

Chapter 1

Notation and Preliminaries

In this chapter, we mention a basic definitions and properties of Bézier curves and Minkowski space. The goal of this chapter is to clarify terms and relations, which we use in this work. The following text is based on the works [Ber87b, Ber87a, DFN91, Cha10].

1.1 Minkowski space

Pseudo-euclidean space, denoted by $\mathbb{R}^n_p$, $n \in \mathbb{N}$, $p \in \mathbb{N}_0$ is an $n-$dimensional real vector space with a regular quadratic form $\mathbf{q}\colon \mathbb{R}^n \to \mathbb{R}$, where $\mathbf{q}(x_1, \ldots, x_n) = \sum_{i=1}^{n-p} x_i^2 - \sum_{j=n-p+1}^{n} x_j^2$ in certain basis. For $p = 1$, we call it *Minkowski space*, for $p = 0$, we get Euclidean space. We use the notation $\overline{x} = (x_1, \ldots, x_n)$ for vectors in $\mathbb{R}^n_p$.

Let $M_{n,n}(\mathbb{R})$ be the set of $n \times n$ matrices with real coefficients. We can write the quadratic form $\mathbf{q}$ in a certain basis of $\mathbb{R}^n$ in the matrix form as $\mathbf{q}(\overline{x}) = \overline{x}^\top Q \overline{x}$, where $Q \in M_{n,n}(\mathbb{R})$ is symmetric and regular. A quadratic form has an associated polar form $P\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by $P(\overline{x}, \overline{y}) = \overline{x}^\top Q \overline{y}$. Clearly, it is bilinear and symmetric.

Since $\mathbf{q}(\overline{x}) = P(\overline{x}, \overline{x})$, the polar form plays a role of scalar product. Hence, we call it *pseudo-scalar product*. We say, the vectors $\overline{x}, \overline{y} \in \mathbb{R}^n_p$ are *pseudo-orthogonal* if $P(\overline{x}, \overline{y}) = 0$.

Although we consider vector space $\mathbb{R}$, we get by standard construction an affine space with a *pseudo-Cartesian* coordinate system $S(O, x_1, \ldots, x_n)$, where the axes $x_1, \ldots, x_n$ are pseudo-orthogonal and O is the origin. We say, that the coordinate axes $x_1, \ldots, x_{n-p}$ are space-like and the axes $x_{n-p+1}, \ldots, x_n$ are time-like. In the following we work using such a setup.

Using the quadratic form, we classify the vectors in the pseudo-euclidean space. We call the vector $\overline{x} \in \mathbb{R}^n_p$ *space-like* if $\mathbf{q}(\overline{x}) > 0$, *time-like* if $\mathbf{q}(\overline{x}) < 0$ and *light-like* if $\mathbf{q}(\overline{x}) = 0$. All the vectors in $\mathbf{q}^{-1}(0)$ are also called *isotropic* and they form a cone Q. The set of all light-like vectors forms the *isotropic cone* Q of the corresponding quadratic form. If a subspace $F \subset \mathbb{R}^n_p$ consists of isotropic vectors, it is called *isotropic subspace*. We say that the coordinate axes $x_1, \ldots, x_{n-p}$ are space-like and the axes $x_{n-p+1}, \ldots, x_n$ are time-like.

A point $x \in \mathbb{R}^n_p$ is space-like (time-like, light-like respectively) if its position vector $\overline{x} = x - O$ is such. There are two possible ways, how to define space-like curve (time-like and light-like respectively).

Definition 1. A differentiable curve $p\colon I \to \mathbb{R}^n_p$ is called *space-like* if the tangent vector $\dot{p}(t)$ is space-like for each $t \in I$.

Definition 2 (Space-like curve)**.** A curve $p\colon I \to \mathbb{R}^n_p$ is called *space-like* if it contains only space-like points, i.e. vector $\overline{p(t)} = p(t) - O$ is space-like vector for every $t \in I$.

Some properties of the curves, which we know from the Euclidean geometry, changes in pseudo-euclidean geometry. Specially, those that depend on the scalar product. For example, the Frenet formulas given in [YT08] are much more complicated and it is necessary to distinguish a type of curve. The works [KJ06, KL10] show that any rational Minkowski Pythagorean hodograph curve can be obtained in terms of its associated planar rational Pythagorean hodograph curve and an additional rational function. The classification of all spacelike curves with constant curvatures in four dimensional Minkowski space can be found in [PŠ02]. The differential geometry of curves and surfaces is covered in [Küh06].

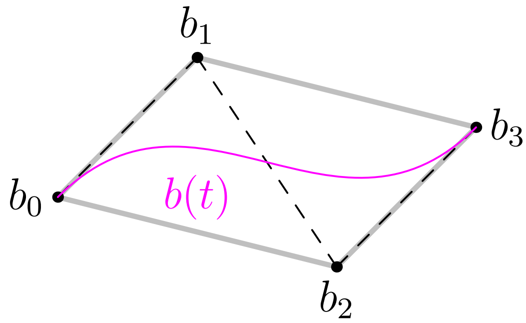


Figure 1.1: The Bézier curve $b(t)$ always passes through the first and the last control point and lies within the convex hull of its control points $b_i \in \mathbb{R}^n_1$.

In our work, we use the definition 2 of space-like curve. One of the advantages is that the condition of differentiability is not required.

Throughout the text, the basic geometric objects are marked as follows. We use the notation $\overleftrightarrow{AB}$ for lines, $\overrightarrow{AB}$ for half-lines and AB for segment. Let t be the line in the plane. It divides the plane into two disjoint half-planes, we denote them H^+_t (open half-plane), $\overline{H^-_t}$ (closed half-plane). In particular situation we describe, which of the half-planes is closed/opened.

We say, that two curves $c_1\colon f(x,y) = 0$ and $c_2\colon x = x(t)$, $y = y(t)$, $t \in I$ have contact of order k, if the derivatives of $f(x(t), y(t))$ in the point $t = 0$ vanishes up to the order $(k-1)$ and the $k-$th derivative is non zero.

1.2 Bézier curves and surfaces

Definition 3 (Bézier curve)**.** *Bézier curve* in Minkowski space of degree n is a polynomial map $b\colon [0,1] \to \mathbb{R}^n_1$ given by $b\left(t\right) = \sum_{i=0}^{n} B^n_i\left(t\right) b_i$. The points $b_i \in \mathbb{R}^n_1$ are called *control points*, $B^n_i\left(t\right) = \binom{n}{i}\left(1-t\right)^{n-i} t^i$ for $i \in \{0, \ldots, n\}$ are Bernstein polynomials of degree n.

The Bézier curve $b(t)$ always passes through the first and the last control point and lies within the convex hull of its control points, see the fig. 1.1. The construction of Bézier curves is invariant under affine transformations, but not invariant under all projective transformations.

Theorem 1. *Let the points $A, B, T \in \mathbb{R}^2_1$ be non collinear and t be a line such that $A, B \notin t$. Then, the quadratic Bézier curve $b(t)$ with end points A, B, containing the point T and with tangent line t in the point T, is uniquely determined.*

Proof. Let the $\overline{u} = (u_1, u_2) \neq (0, 0)$ be the direction vector of the tangent line t and $A = [a_x, a_y], B = [b_x, b_y], T = [t_x, t_y]$. We can define the map τ such that $\tau(A, B, T, \overline{u}) = C$ returns the middle control point of the Bézier curve $b(t)$. By using the equations

$$b(t_0) = T = B^2_0(t_0)A + B^2_1(t_0)C + B^2_2(t_0)B \ ,$$
$$\dot{b}(t_0) = k\overline{u} \ ,$$

for $C \in \mathbb{R}^2_1$, $k \neq 0 \in \mathbb{R}$, $t_0 \in (0, 1)$ as unknowns, we obtain

$$C = \frac{T - B^2_0(t_0)A - B^2_2(t_0)B}{B^2_1(t_0)} \ ,$$
$$u_1(-2(1-t_0)a_y + 2(1-2t_0)c_y + 2t_0b_y) = u_2(-2(1-t_0)a_x + 2(1-2t_0)c_x + 2t_0b_x) \ ,$$

and if we substitute c_x, c_y using the first equation, we obtain

$$\tau(A, B, T, \overline{u}) = C = \frac{T - B^2_0(t_0)A - B^2_2(t_0)B}{B^2_1(t_0)} \ , \tag{1.1}$$

where $t_0 \in (0, 1)$ is a solution of the equation

$$0 = \alpha t_0^2 + \beta t_0 + \gamma \ , \text{ where}$$
$$\alpha = 2\left(u_1(b_y - a_y) + u_2(a_x - b_x)\right) \ ,$$
$$\beta = 4\left(u_1(a_y - t_y) - u_2(a_x - t_x)\right) \ ,$$
$$\gamma = -\frac{\beta}{2} \ .$$

The Bézier curve $b_{ACB}(t)$ with control points A, C, B in this order satisfies the requirements of the theorem. □

Some properties of Bézier curves in $\mathbb{R}^3_1$ are proved in the paper [Geo08]. The author use the definition 1 of space-like curves. Let us mention some of them.

Theorem 2. *Let* $b_M(t) = \sum_{i=0}^{n} B_i^n(t)\, b_i$ *be the Bézier curve in Minkowski space. If the vectors* $\triangle b_i = b_{i+1} - b_i$ *for* $i \in \{0, \ldots, n-1\}$ *are space-like, then the curve* $b_M(t)$ *is space-like.*

Theorem 3. *Let* $\pi_i \colon \mathbb{R}_1^3 \to \mathbb{R}$, *where* $i = 1, 2, 3$ *is such that for* $\overline{x} = (x_1, x_2, x_3) \in \mathbb{R}_1^3$ *holds* $\pi_i(\overline{x}) = x_i$. *Let* $b_M(t)$ *is Bézier curve in* $\mathbb{R}_1^3$. *If* $\pi_1(\triangle b_i) = \pi_2(\triangle b_i)$ *or* $\pi_1(\triangle b_i) = \pi_3(\triangle b_i)$ *for* $i \in \{0, \ldots, n-1\}$, *then* $b_M(t)$ *is space-like and non regular.*

Theorem 4. *If the Bézier curve* $b_M(t)$ *is space-like, then the vectors* $\triangle b_0 = b_1 - b_0$ *and* $\triangle b_{n-1} = b_n - b_{n-1}$ *are space-like.*

Note 1. The space-like Bézier curve $b_M(t)$ with control points $b_0, \ldots, b_n$, where $n > 3$ is closed, if $b_0 = b_n$ and b_0 is in the middle of the segment $[b_1, b_{n-1}]$.

Definition 4 (Pseudo-vector product)**.** The *pseudo-vector product* is bilinear function $\theta \colon \mathbb{R}_1^3 \times \mathbb{R}_1^3 \to \mathbb{R}_1^3$ such that for given $\overline{x} = (x_1, x_2, x_3), \overline{y} = (y_1, y_2, y_3) \in \mathbb{R}_1^3$ is $\theta(\overline{x}, \overline{y}) = \overline{x} \times \overline{y} = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, -x_1 y_2 + x_2 y_1)$.

Theorem 5. *Let* $b_M(t)$ *be space-like Bézier curve. Let us define the functions*

$$
\begin{aligned}
Q_1(t) &= \langle b_M'(t), b_M'(t) \rangle, \\
Q_2(t) &= \langle b_M'(t) \times b_M''(t), b_M'(t) \times b_M''(t) \rangle, \\
Q_3(t) &= \langle b_M'(t) \times b_M''(t), b_M'''(t) \rangle .
\end{aligned}
$$

If $b_M(t)$ *is regular for some* $t_0 \in [0, 1]$, *then for the curvature and torsion of Bézier curve in the point* t_0 *holds*

$$
\begin{aligned}
\kappa(t_0) &= \frac{\sqrt{|Q_2(t_0)|}}{\left(\sqrt{Q_1(t_0)}\right)^3}, \\
\tau(t_0) &= \frac{Q_3(t_0)}{Q_2(t_0)} .
\end{aligned}
$$

There are also articles about the surfaces, which can be expressed in Bézier form. The properties of space-like Bézier surfaces in $\mathbb{R}_1^3$ are proved in the papers [Geo09] and [UMY11]. The author of [Geo09] obtain sufficient conditions for Bézier surfaces to be space-like. The authors of [UMY11] give the conditions of the time-like case and the space-like case for Bézier surfaces.

Let the surface S be given with a parameterization,

$$\phi\colon U \subset \mathbb{R}^2 \to \mathbb{R}^3_1$$
$$(u, v) \to \phi(u, v) = (\phi_1(u, v), \phi_2(u, v), \phi_3(u, v)) \ ,$$

where ϕ is smoothly differentiable in U. Assume that the vectors $\phi_u = \frac{\partial}{\partial u}\phi(u, v)$ and $\phi_v = \frac{\partial}{\partial v}\phi(u, v)$ are linearly independent for any $(u, v) \in U$. Then S is called *a regular surface* in $\mathbb{R}^3_1$.
The normal vector field N on S is given by $N = \phi_u \times \phi_v$ and N is not necessarily unit vector field. The tangent space $T_P(S)$ to S at the point $P = \phi(u, v)$ is two dimensional subspace of $\mathbb{R}^3_1$, which is spanned by the vectors ϕ_u and ϕ_v.

Definition 5 (Time-like, space-like surface)**.** The surface $S \subset \mathbb{R}^3_1$ is called *space-like* if for any $(u, v) \in U$ both vectors ϕ_u and ϕ_v are space-like. This is equivalent to saying that the each vector in normal vector field N is time-like.
The surface S is called *time-like* if the normal vector field N consists only from space-like vectors.

In [Geo09], we can find the following definition and proposition.

Definition 6 (Tensor-product Bézier surface)**.** Let the points P_{ij} for $i \in \{0, \ldots, m\}$, $j \in \{0, \ldots, n\}$ be points in $\mathbb{R}^3_1$ and $\overline{p}_{ij}$ be their position vectors. A polynomial surface with a parametric equation $B_{ten}(u, v) = \sum_{i=0}^{m}\sum_{j=0}^{n} B_i^m(u)\, B_j^n(v)\, \overline{p}_{ij}$, where $u \in [0, 1]$, $v \in [0, 1]$, is called a *tensor-product Bézier surface* in $\mathbb{R}^3_1$ with control points P_{ij}.

Theorem 6. *Let $B_{ten}(u, v)$ be the tensor-product Bézier surface in $\mathbb{R}^3_1$. Suppose that $\triangle_{i,j} = \overline{p}_{i+1,j} - \overline{p}_{i,j}$ and $\overline{\triangle}_{i,j} = \overline{p}_{i,j+1} - \overline{p}_{i,j}$ for $i \in \{0, \ldots, m-1\}$, $j \in \{0, \ldots, n-1\}$. If the vectors $\triangle_{i,0} = \triangle_{i,1} = \ldots = \triangle_{i,n}$ and $\overline{\triangle}_{0,j}$ for $i \in \{0, \ldots, m-1\}$, $j \in \{0, \ldots, n-1\}$ are space-like, then the surface $B_{ten}(u, v)$ is space-like.*

The paper [UMY11] shows that type of surface depends on the corresponding control net of the surface.

The polar form P in $\mathbb{R}^3_1 \supset S$ induces the polar form denoted by P_P in each tangent plane $T_P(S)$ of the smooth surface S. For any $\overline{x}, \overline{y} \in T_P(S) \subset \mathbb{R}^3_1$ holds $P(\overline{x}, \overline{y}) = P_P(\overline{x}, \overline{y})$. The symmetric, bilinear polar form P_P has an associated quadratic form $I_P \colon T_P(S) \to \mathbb{R}$ given by $I_P(\overline{w}) = P_P(\overline{w}, \overline{w})$. The quadratic form I_P is called *the first fundamental form* of the surface S in the point $P \in S$. The first fundamental form of the surface S can be expressed in the basis $\{\phi_u, \phi_v\}$ as $I_P(\overline{w}) = Edu^2 + Fdudv + Gdv^2$, where $E(u,v) = P_P(\phi_u, \phi_u), F(u,v) = P_P(\phi_u, \phi_v), G(u,v) = P_P(\phi_v, \phi_v)$ are the differentiable coefficients. In the paper, the first fundamental form coefficients are derived in terms of coordinates of control points of the surface, we denote them E_B, F_B, G_B. Then, the next proposition holds.

Theorem 7. *Let $B(u,v)$ be a Bézier surface in $\mathbb{R}^3_1$ three-dimensional Minkowski space. For $(u,v) \in [0,1] \times [0,1]$, $B(u,v)$ is called a time-like (resp. space-like) surface if $F_B^2 - E_B G_B > 0$ (resp. $F_B^2 - E_B G_B < 0$).*

The authors study the Plateau problem ([Mon03, Mon04]) in time-like and space-like Bézier surfaces using the extremal of the Dirichlet functional in $\mathbb{R}^3_1$. There are given some examples for these cases, and they compare the area functionals for the minimal Bézier surface in $\mathbb{R}^3$ and $\mathbb{R}^3_1$.

1.3 Extended Euclidean plane

Let E^2 be an Euclidean plane. We assign to each class of parallel lines a unique *point at infinity*, at which all of the lines meet. All the points at infinity define the *line at infinity* l^∞. The *extended Euclidean plane*, denoted by $\overline{E^2}$, is obtained as $\overline{E^2} = E^2 \cup l^\infty$.

So we extend the set of all real points in E^2 with infinite number of points at infinity, but we extend the set of all real lines in E^2 with only one line at infinity (l^∞). In extended Euclidean plane, any two distinct points determine a unique line (collinearity) and any two distinct lines cross at a unique point (concurrence).

These statements are *dual*, which means they are formed by changing points to lines and collinearity to concurrence.

Let $S(O, e_1, e_2)$ be an affine coordinate system in E^2. Then each point at infinity $a^\infty \in \overline{E^2}$ corresponding to real line a has homogeneous coordinates $[x_1, x_2, 0]$, where $x_1, x_2 \in \mathbb{R}$ and $(x_1, x_2) \neq (0, 0)$ is direction vector of the real line a. Each real point A belongs to real line a has homogeneous coordinates $[x_1, x_2, x_3]$, where $x_3 \neq 0$ and $[\frac{x_1}{x_3}, \frac{x_2}{x_3}]$ are its affine coordinates. Hence, each point in $\overline{E^2}$ (real or at infinity) has an infinite number of triplet of homogeneous coordinates.

1.4 Conic sections

A conic section K in $\mathbb{R}^3_1$ is a curve obtained by intersecting a cone (more precisely, a right circular conical surface) with a plane. This plane algebraic curve of degree 2 is the set $K = \{[x, y] \in \mathbb{R}^2 \colon f(x, y) = 0\}$, where $f(x, y) = k_A x^2 + 2k_B xy + k_C y^2 + 2k_D x + 2k_E y + k_F$. In appropriate cases, we consider the equation of the conic section instead of K due to the fact the field $\mathbb{R}$ is not algebraically closed.

We can write the equation of a conic section in the matrix form as

$$f(x, y) = \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} k_A & k_B & k_D \\ k_B & k_C & k_E \\ k_D & k_E & k_F \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}, \tag{1.2}$$

where $(x, y, 1)$ are extended coordinates of point (x, y) in $\overline{E^2}$. We denote this 3×3 matrix by M_K.

The determinant $\Delta = \det(M_K)$ is called the *determinant of the conic section.* And the determinant $\delta = \det\left(\left(\begin{smallmatrix} k_A & k_B \\ k_B & k_C \end{smallmatrix}\right)\right)$ is called the *discriminant of the conic section.*

If $\Delta \neq 0$, the conic section is regular. If $\delta = 0$ then the conic section is a parabola, if $\delta < 0$, it is an hyperbola and if $\delta > 0$, it is an ellipse. A conic section is a circle if $\delta > 0$ and $k_A = k_C$ and $k_B = 0$.

If $\Delta = 0$, the conic is a degenerate parabola (two coinciding lines), a degenerate ellipse (a point ellipse), or a degenerate hyperbola (two intersecting lines).

In the extended Euclidean plane, it is necessary to homogenize the equation (1.2) by replacing $(x, y, 1)$ with (x, y, z). We obtain the conic section $K = \{[x, y, z] \in \overline{\mathbb{R}^2} \colon f(x, y, z) = 0\}$ in homogeneous coordinates.

From each point $X \in \mathbb{R}^2$ lying outside the regular conic section, one can construct two tangent lines to regular K. The corresponding points of contact may be either *real* or *at infinity*. In the case of point at infinity of the contact, the conic section K is hyperbola and the tangent line is the asymptote a. We denote by a^∞ the point of contact at infinity, see fig. 3.3. We denote the set of all tangent lines to K by T_K. We denote by $\nabla f(x_0, y_0)$ the gradient of K in the point $[x_0, y_0] \in K$.

Chapter 2

Space-like conditions for quadratic Bézier curves

Let us consider a Minkowski space $\mathbb{R}_1^3$ and a quadratic Bézier curve with control points A, C, B in this order, denoted by $b_{ACB}(t)$. We find the necessary and sufficient condition for the control points A, B. Then, we fix them and we are looking for the set of all such points C that the Bézier curve $b_{ACB}(t)$ is space-like.

2.1 Necessary conditions for the end points

In order a Bézier curve to be space-like, each of its points have to be space-like. Let the points $A = [a_1, a_2, a_3]$ and $B = [b_1, b_2, b_3]$ be fixed. Since Bézier curve interpolates its endpoints, we have necessary and sufficient conditions

$$a_1^2 + a_2^2 - a_3^2 > 0 \ , \tag{2.1}$$

$$b_1^2 + b_2^2 - b_3^2 > 0 \ . \tag{2.2}$$

A generic quadratic Bézier curve is a part of a parabola (see the fig. 2.2), so it lies in the plane $\rho \subset \mathbb{R}_1^3$. Since the given points $A, B \in \rho$, the construction of the plane ρ has several degrees of freedom depending on their positions. By using

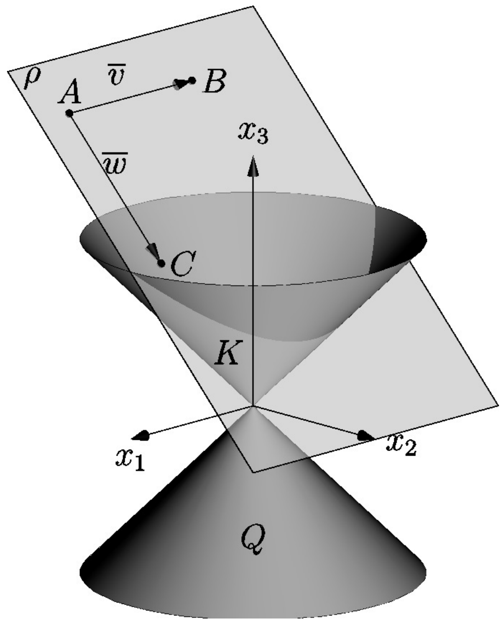


Figure 2.1: Plane ρ spans points A, B, C. In case of their non-collinearity, they generate ρ as affine hull. The conic section K is an intersection of the light cone Q and the plane ρ.

the equation $\rho = \{X \in \mathbb{R}^3_1;\ X = A + t\overline{v} + s\overline{w},\ \text{for } t, s \in \mathbb{R}\}$, the degree of freedom is represented by the vectors $\overline{v}, \overline{w}$. If $A \neq B$, the degree of freedom is 1. As the position of the point C changes, the plane ρ eventually rotates about the axis $\overleftrightarrow{AB}$, so we can choose as $\overline{v} = B - A$ and the choice of the vector $\overline{w}$ is free. If $A = B$, the degree of freedom is 2 and the choice of both vectors $\overline{v}, \overline{w}$ is free.
In any case, the intersection of the light-cone Q and the plane ρ is a conic section K (see fig. 2.1). The figure 2.3 shows how the set of all space-like points S in the plane ρ looks like. The space-like Bézier curve $b_{ACB}(t)$ lies outside the light-cone Q, hence outside of the conic section K in the plane ρ ($b_{ACB}(t) \subset S$). We present a solution in the plane ρ for each type of conic section and the planar results can be put together to form the spatial result.

As we shall see later, it is useful to consider ρ as an extended pseudo-euclidean plane. But we consider only real points $A, B, C \notin l^\infty$. Let $S_\rho(O, x, y)$ be a pseudo-Cartesian coordinate system in the plane ρ. Let $A = [a_x, a_y]$, $C = [c_x, c_y]$ a $B = [b_x, b_y]$ be the local affine coordinates of the control points in $S_\rho(O, x, y)$.

From now, the points A, B are fixed and they satisfy the conditions (2.1), (2.2).

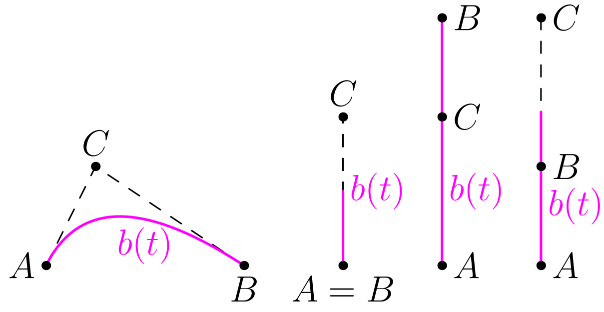


Figure 2.2: Types of quadratic Bézier curves.

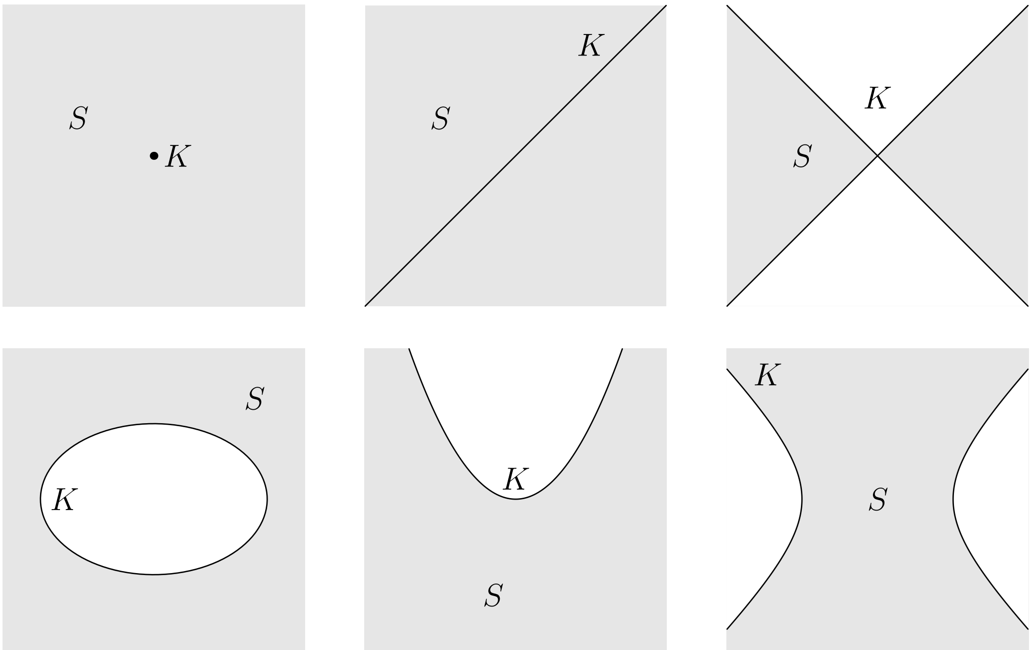


Figure 2.3: Let $K \subset \rho$ be the conic section (point, double line, pair of lines, ellipse, parabola, hyperbola). The set S consists of all space-like points in the plane ρ.

Definition 7 (Set of admissible solutions)**.** Let $V_\rho(A, B)$ be a set of points $C \in \rho$ such that the curve b_{ACB} is space-like. Then, we say that $V_\rho(A, B)$ is a *set of admissible solutions*.

Chapter 3

Set of admissible points of contact

In this chapter, we study for the given points A, B all Bézier curves b_{ACB} such that common points of the curve b_{ACB} and conic section K are only the points of their contact. It is natural, because a "boundary" between the situation that two curves have no common points and the situation that one curve intersects the other curve is, that they touch each other.

Definition 8 (Set of points of contact)**.** We say that the set $D \subset K$ is the *set of points of contact* between K and the set of all b_{ACB} if for any point $X = [x_0, y_0] \in D$ there is a point C such that $b_{ACB}(t) \cap K = M$, where $X \in M$ is a point of contact of order 2 and the set M contains only points of contact of order 2 between $b_{ACB}(t)$ and K.

Note 2. The set M contains at most two points, since two different quadratic curves may have at most two common points of contact of order 2 (Bézout theorem [Kun05]). If we will mark a point by letter T, we mean a point of the contact.

Definition 9 (Double contact)**.** We say that b_{ACB} has *double contact*, if it has with K exactly two points of contact of order 2 (i.e. the set M contains exactly two points). We denote the middle control point of the Bézier curve C_u. If K is regular conic section, we denote the touch points by the letter U_i (see the fig. 6.4). If K is singular conic section, we denote the touch points by the letter S_i (see the fig. 6.2(b)).

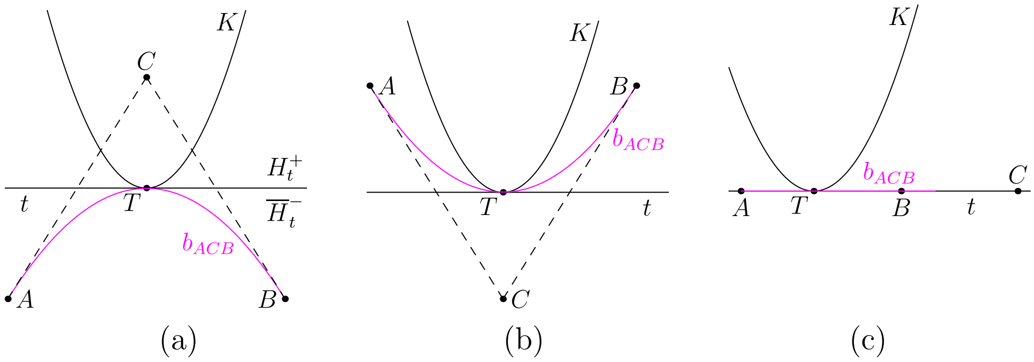


Figure 3.1: (a) exterior point of contact T (b) interior point of contact T (c) exterior point of contact T

When we obtain K as a connected component of regular conic section, it may be ellipse, parabola or one component of hyperbola.

Definition 10 (Separating tangent line)**.** Let t be the tangent line to connected component of regular conic section K in the point $T \in K$. It divides the plane ρ into two disjoint half-planes $H_t^+, \overline{H}_t^-$ such that $K \setminus \{T\} \subset H_t^+$. We say, that t *separates* the connected component of conic section K and the set of points O, if they lie in the different half-planes with respect to the tangent t. I.e. $K \setminus \{T\} \subset H_t^+$ and $O \subset \overline{H}_t^-$ (see fig. 3.1). We denote the set of all separating tangent lines by $T_{sep}(O, K)$. The corresponding points of contact are real or points at infinity. We denote by $S(O, K) \subset K$ the set of all *real* points of contact.

Definition 11 (Exterior (interior) point of contact)**.** We say that $b_{ACB}\,(t)$ *touches* connected component of regular conic section K from the *outside (inside)*, if their common tangent is (is not) separating (see the fig. 3.1). Then, the point of contact is called *exterior (interior) point of contact.*

Note 3. The set of all exterior (interior) points of contact is denoted D_{ext} (D_{in}). In the case of regular conic section, the set of points of contact $D = D_{ext} \cup D_{in}$.

Now, we describe the set of points of contact D for every type of conic section.

Let K be a singular conic section. In the case of $K = \{V_Q\}$, where V_Q is the top of the isotropic cone Q, we have $D = \{V_Q\}$ (see the figure 6.1).

If $K = p$, where p is an isotropic double line, we must distinguish two cases. If A, B

lie in the opposite half-planes generated by the line p, we have $D = \emptyset$. If A, B lie in the same half-plane, the set of points of contact $D = p$ (see the figure 6.2(a)).
The last singular case is $K = p \cup r$, where p, r are pair of distinct isotropic lines. Then, there are two space-like regions in the plane ρ. If A, B lie in the different regions, there is no space-like Bézier curve b_{ACB}. Let A, B lie in the same region, which is determined by two half-lines $\overrightarrow{V_Q P} \subset p$ and $\overrightarrow{V_Q R} \subset r$. Let $S_p \in p$ and $S_r \in r$ are the points of contact of the Bézier curve b_{AC_uB}, i.e. $b_{AC_uB} \cap K = \{S_p, S_r\}$ is double contact. Then $D = \overrightarrow{S_p P} \cup \overrightarrow{S_r R}$ (see the figure 6.2(b)). The special case is $S_p = S_r = V_Q$.

Let K be a regular conic section. From each space-like point $X \in \rho$, one can construct two tangent lines to K.

3.1 Set of exterior points of contact

Now, we describe the set $T_{sep}(AB, K)$ for the given A, B, because the corresponding real points of contact $S(AB, K)$ form the set D_{ext}.

Let K be a regular conic section, but not hyperbola. Let $t_{1A}, t_{2A}, t_{1B}, t_{2B}$ be the tangent lines from the points A, B to K and the points $T_{1A}, T_{2A}, T_{1B}, T_{2B} \in K$ be the corresponding real points of contact. The points T_{1A}, T_{2A} split the conic section K to some arcs. We denote $\overset{\frown}{T_{1A}T_{2A}}$ the arc $\overset{\frown}{T_{1A}T_{2A}} \subset \triangle AT_{1A}T_{2A}$. Similarly, for points B, T_{1B}, T_{2B} we get $\overset{\frown}{T_{1B}T_{2B}}$. In order A (B) to be separated from K, the point $T \in \overset{\frown}{T_{1A}T_{2A}}$ $(T \in \overset{\frown}{T_{1B}T_{2B}})$. So, the set $S(A, K) = \overset{\frown}{T_{1A}T_{2A}}$ and $S(B, K) = \overset{\frown}{T_{1B}T_{2B}}$ (see fig. 3.2).

If $K = K_1 \cup K_2$ is hyperbola, where K_1, K_2 are connected components of K, we need to consider each of them separately. Let a_1, a_2 be the asymptotes of hyperbola. It may not hold that all the points of contact $T_{1A}, T_{2A}, T_{1B}, T_{2B} \in \{K_1 \cup a_1^\infty \cup a_2^\infty\}$. Let both $T_{1A}, T_{2A} \in K_2$. If $t \in T_{K_1}$, then also $t \in T_{sep}(A, K_1)$. Hence, if we analyze the case of K_1, then $S(A, K_1) = \overset{\frown}{a_1^\infty a_2^\infty} = K_1$ and $T_{sep}(A, K_1) = T_{K_1}$. Let $T_{1A} \in \{K_1 \cup a_1^\infty \cup a_2^\infty\}$ and $T_{2A} \in K_2$. Since the asymptote $a_1 \in T_{sep}(A, K_1)$, the set

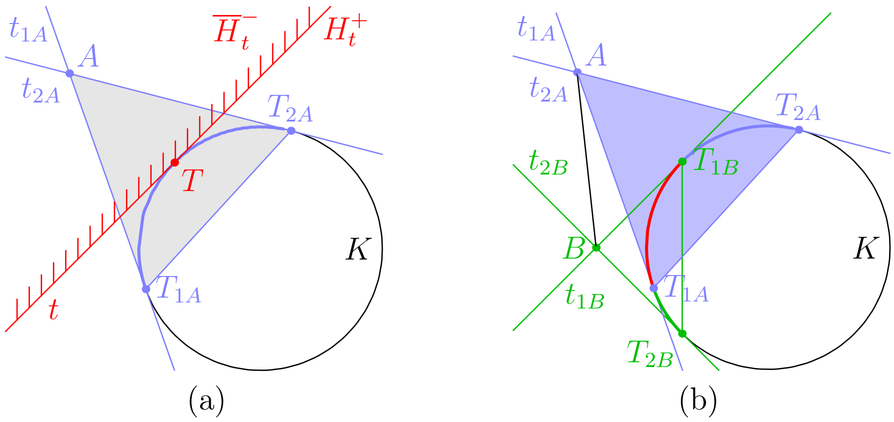


Figure 3.2: (a) the arc $S(A, K) = \overset{\frown}{T_{1A}T_{2A}} \subset \triangle AT_{1A}T_{2A}$ determines all tangent lines $T_{sep}(A, K)$ that separate A and K
(b) the arc $S(A, K) \cap S(B, K)$ determines all tangent lines $T_{sep}(AB, K)$ that separate AB and K

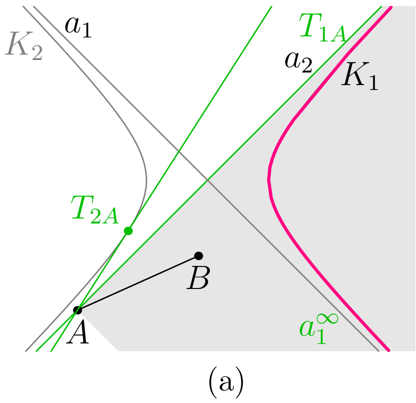
$S(A, K_1) = \overset{\frown}{T_{1A}a_1^{\infty}} \subset K_1$ define the set $T_{sep}(A, K_1)$. Similarly, for the point B or K_2. For the corresponding illustration see fig. 3.3.

Lemma 8 (Separating tangent line)**.** *Let A, B be space-like points and K be a connected component of regular conic section. Let $t \in T_K$ and the point $T \in K$ be its point of contact. The tangent line $t \in T_{sep}(AB, K)$ if and only if $T \in S(A, K) \cap S(B, K)$ (see fig. 3.2 for the case of ellipse).*

Proof. Sufficient condition. If the tangent line t is separating, then the points A, B and the conic section K are separated. In order A (B) to be separated from K, the point $T \in S(A, K)$ ($T \in S(B, K)$). Since A and B should be separated simultaneously, T must belong to the intersection of these two arcs.

Necessary condition. Let a point $T \in S(A, K) \cap S(B, K)$ and $K \setminus \{T\} \subset H_t^+$. Because the point $T \in S(A, K)$ we have the point $A \in \overline{H}_t^-$. Also the point $T \in S(B, K)$, so the point $B \in \overline{H}_t^-$. Therefore, entire segment $AB \subset \overline{H}_t^-$ due to convexity of $\overline{H}_t^-$. □

Theorem 9 (Set of exterior points of contact)**.** *Let A, B be space-like points and K be a regular connected conic section. The set of exterior points of contact $D_{ext} \neq \emptyset$ if and only if the segment $AB \cap K = \emptyset$ or $AB \cap K = \{T_0\}$.*



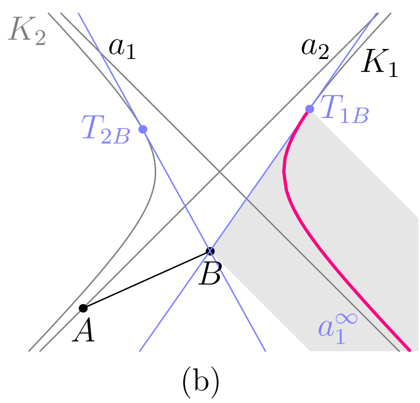


Figure 3.3: (a) Let us find, how the set $T_{sep}(A, K_1)$ looks like. The point $T_{1A} = a_2^\infty \in \{K_1 \cup a_1^\infty \cup a_2^\infty\}$. Because $T_{2A} \in K_2$, we have to "replace it" by $a_1^\infty \in \{K_1 \cup a_1^\infty \cup a_2^\infty\}$, where the asymptote $a_1 \in T_{sep}(A, K_1)$. The set of separating tangent lines $T_{sep}(A, K_1)$ is determined by the arc of corresponding points of contact $S(A, K_1) = \overset{\frown}{T_{1A}a_1^\infty} \subset \triangle AT_{1A}a_1^\infty$. Since $T_{1A} = a_2^\infty$, we have $S(A, K_1) = K_1$.
(b) If we are seeking the set $T_{sep}(B, K_1)$, we have to "replace" the $T_{2B} \in K_2$ by $a_1^\infty \in \{K_1 \cup a_1^\infty \cup a_2^\infty\}$ (not by a_2^∞, because $a_2 \notin T_{sep}(B, K_1)$). The set $T_{sep}(B, K_1)$ is determined by the arc of corresponding points of contact $S(B, K_1) = \overset{\frown}{T_{1B}a_1^\infty} \subset \triangle BT_{1B}a_1^\infty$.

The set D_{ext} consists of an arc $S(A, K) \cap S(B, K)$ on K. The end point of the arc $T \in D_{ext}$ if and only if its corresponding tangent line t to K contains both the end points of the curve $b_{ACB}(t)$, the control points A and B.

Note 4. As we said, we consider the hyperbola as two separate subsets K_1, K_2 of a conic section. Therefore, there could be two continuous arcs D_{ext}^1, D_{ext}^2, one on each component K_1, K_2. Then $D_{ext} = D_{ext}^1 \cup D_{ext}^2$.

Proof. Sufficient condition. Let $AB \cap K = \{X_1, X_2\}$. Then $S(A, K) \cap S(B, K) = \emptyset$ so there exists no separating tangent line $t \in T_{sep}(AB, K)$. Hence, for any Bézier curve b_{ACB} there exists no separating tangent line $T_{sep}(b_{ACB}, K)$. Consequently the set $D_{ext} = \emptyset$.

Necessary condition. At first, let $AB \cap K = \emptyset$. Then $S(A, K) \cap S(B, K) \neq \emptyset$. Let $t \in T_K$ and the point $T \in K$ be its corresponding point of contact. Using the theorem 1, the quadratic curve b_{ACB} is uniquely determined by the points A, B, T and by the tangent line t. If $T \notin S(A, K) \cap S(B, K)$, then the tangent line $t \notin T_{sep}(AB, K)$, $t \notin T_{sep}(b_{ACB}, K)$ and $T \notin D_{ext}$. On the other hand, because quadratic Bézier

curve is a convex curve, each of its tangent line defines the supporting half-plane to the curve. If $t \in T_{sep}(A, K)$ and $t \in T_{sep}(B, K)$, then $t \in T_{sep}(b_{ACB}, K)$. So, for every $T \in S(A, K) \cap S(B, K)$ holds that $t \in T_{sep}(b_{ACB}, K)$. Hence, we have $D_{ext} = S(A, K) \cap S(B, K) \neq \emptyset$.
Now, let $AB \cap K = \{T_0\}$, then $S(A, K) \cap S(B, K) = \{T_0\}$. So there is only one separating tangent line $t_0 \in T_{sep}(AB, K)$ and $D_{ext} = \{T_0\} \neq \emptyset$.

Without loss of generality, let Bézier curve b_{ACB} is determined by the points A, B, T_{1A} and tangent line t_{1A} to K and $T_{1A} \in D_{ext}$. It holds if and only if the tangent line to b_{ACB} in the end point A is the same as tangent line t_{1A}. It holds if and only if the Bézier curve is a segment and the points $A, B, T_{1A} \in t_{1A}$ are collinear. Note: if for both end points T_1, T_2 of the arc D_{ext} holds that $T_1, T_2 \in D_{ext}$, then $A = B$. □

Note 5. We denote by D_e the set of all $T \in D_{ext}$ such that the corresponding tangent line t to K at T contains both control points A, B.

3.2 Set of interior points of contact

Lemma 10. *If $T \in D_{in} \subset D$ is an interior point of contact of the Bézier curve $b(t)$ and the conic section K, then there exists the triangle $\triangle ABC$ such that $T \in \triangle ABC$ and the sides AC, CB have no common point with the conic section K, see fig. 3.4.*

Proof. Let the Bézier curve b_{ACB} be determined by the end points A, B and the tangent line t at the point $T \in K \cap b_{ACB}$. Then, the triangle $\triangle ABC$ determined by the control points of the Bézier curve satisfies the requirements of the lemma. The point $T \in \triangle ABC$ because the Bézier curve is completely contained in the convex hull of its control points. The sides AC, CB have no common point with the conic section K because they are separated from K by the curve b_{ACB}. □

The line $\overleftrightarrow{AB}$ divides the plane ρ into two half-planes, the open half-plane H^-_{AB}, and the closed half-plane $\overline{H^+_{AB}}$ such that $\overleftrightarrow{AB} \subset \overline{H^+_{AB}}$. Let us divide tangent lines

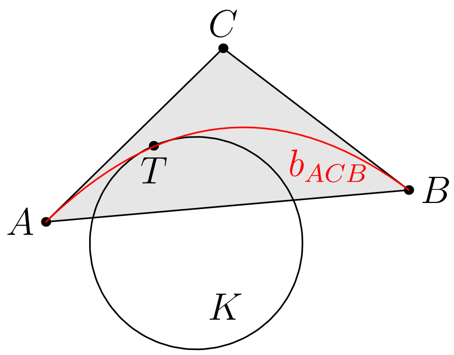


Figure 3.4: For the point $T \in D_{in} \subset D$ there exists the triangle $\triangle ABC$ such that $T \in \triangle ABC$ and the sides AC, CB have no common point with the conic section K.

from A, B, that are not in $T_{sep}(AB, K)$, into two pairs t_1^+, t_2^+ and t_1^-, t_2^- such that the corresponding points of contact $T_i^{+,-} = t_i^{+,-} \cap K, i = 1, 2$ lie in the same half-plane $T_{1,2}^+ \in \overline{H_{AB}^+}$ and $T_{1,2}^- \in H_{AB}^-$. See figure 6.5.

Definition 12 (Converging (diverging) tangent lines)**.** If $t_1^+ \cap t_2^+ = P^+ \in \overline{H_{AB}^+}$, then we say that tangents t_1^+, t_2^+ *converge*. If $P^+ \in H_{AB}^-$, then we say they *diverge*. See figure 6.5.

The same definition holds for the pair t_1^-, t_2^-. Let us divide the set D_{in} into the interior points of contact which are located in the half-plane $\overline{H_{AB}^+}$ (resp. H_{AB}^-) denoted by D_{in}^+ (resp. D_{in}^-). There holds $D_{in} = D_{in}^+ \cup D_{in}^-$.

Theorem 11 (Set of interior points of contact)**.** *If the set $D_{in}^+ \neq \emptyset$ ($D_{in}^- \neq \emptyset$), then the pair of tangents t_1^+, t_2^+ (t_1^-, t_2^-) converge.*

Proof. If the mentioned pair of tangent lines diverge, then there exists no triangle $\triangle ABC$ from the lemma 10. □

Hypothesis 1. The condition in the theorem 11 is also necessary for $D_{in} \neq \emptyset$.

Hypothesis 2. If hypothesis 1 holds, let $t_1, t_2 \notin T_{sep}(AB, K)$ be a pair of converging tangent lines. If there exists a point C such that b_{ACB} has double contact and C and $P = t_1 \cap t_2$ lie in the same half-plane with respect to the $\overleftrightarrow{T_1T_2}$, then let us denote the corresponding points of contact U_1, U_2. The set of interior points of contact D_{in} is either the continuous arc $\overset{\frown}{T_1T_2}$ on K or the union of the arcs $\overset{\frown}{T_1U_1} \cup \overset{\frown}{U_2T_2}$. The points $T_1, T_2 \notin D_{in}$, the points $U_1, U_2 \in D_{in}$.

Note 6. If D_{in} contains a set $\{\widehat{T_1U_1} \cup \widehat{U_2T_2}\}$, we will call the union of two continuous arcs $\widehat{T_1U_1}$ and $\widehat{U_2T_2}$ by *sharp arc* (see fig. 6.4). Later, we will see that the union of these two arcs will have the same properties as a continuous arc of interior touches $\widehat{T_1T_2}$. So this sharp is just apparent and this union has structural properties for our use as a continuous arc.

3.3 Set of points of contact

As we said, the set of all points of contact $D = D_{ext} \cup D_{in}$. The following theorem describes the set D for different regular types of conic section K.

Theorem 12 (Set of points of contact)**.**

(a) Let K be an ellipse. Then, the set of the points of contact D is either one arc of the exterior points of contact or one arc of exterior and one arc of interior points of contact or one or two arcs of interior points of contact.

(b) Let K be a parabola. Then, the set of the points of contact D is either one arc of the exterior points of contact or one arc of interior points of contact.

(c) Let K be a hyperbola. Then, the set of the points of contact D is either two arcs of the exterior points of contact or one arc of exterior and one arc of interior points of contact.

The arcs of the interior points of contact may be sharp. The set of exterior points of contact may contains only one point T_0, when segment $AB \cap K = \{T_0\}$.

Proof. a) (Ellipse) Let $AB \cap K = \emptyset$. Then among all tangents from A and B to K, there are two in the set $T_{sep}(AB, K)$. They determine one arc D_{ext}. The other two tangents are not in the set $T_{sep}(AB, K)$. If they converge, then there is also one arc D_{in}. If they diverge, $D_{in} = \emptyset$. If $AB \cap K = \{T_0\}$, the only difference is that from separating pair of tangent lines become one line $\overleftrightarrow{AB}$ and $D_{ext} = \{T_0\}$.
Let $AB \cap K = \{X_1, X_2\}$. Then, we consider two pairs of tangent lines t_1^+, t_2^+ and t_1^-, t_2^-. One pair, without loss of generality the pair t_1^+, t_2^+, always converge

so $D_{in}^{+} \neq \emptyset$. The pair t_1^-, t_2^- may converge or diverge, so we can obtain D as one or two arcs of D_{in}.

b) (Parabola) Let $AB \cap K = \emptyset$. Then among all tangents from A and B to K, there are two in the set $T_{sep}(AB, K)$. They determine one arc D_{ext}. The other two tangents are not in the set $T_{sep}(AB, K)$, but they always diverge so $D_{in} = \emptyset$. Hence, we obtain D as one arc D_{ext}. If $AB \cap K = \{T_0\}$, then $D = D_{ext} = \{T_0\}$.
Let $AB \cap K = \{X_1, X_2\}$. Without loss of generality, the pair t_1^+, t_2^+ always converge so $D_{in}^+ \neq \emptyset$ and the pair t_1^-, t_2^- always diverge so $D_{in}^- = \emptyset$. Hence, we obtain D as one arc D_{in}.

c) (Hyperbola) Each component K_1, K_2 of a hyperbola K separately behaves similarly to the parabola case. There are two cases of configuration AB and K_1, K_2. The first, $AB \cap K_1 = \emptyset$ and also $AB \cap K_2 = \emptyset$. Then, we obtain D as two arcs of D_{ext}. If $AB \cap K_1 = \{T_0\}$, the only difference is that $D_{ext}^1 = \{T_0\}$ (similarly for K_2 if $AB \cap K_2 = \{T_0\}$).
The second case, $AB \cap K_1 = \emptyset$ and $AB \cap K_2 = \{X_1, X_2\}$. Then we obtain D as one arc of $D_{ext} \subset K_1$ and one arc of $D_{in} \subset K_2$. There are no other configurations. $\square$

In the tables 3.1 and 3.2, we see the structure of the set D looks like for various types of conic sections. The numbers in the table 3.2 represent the number of arcs in the set D. The generic cases for regular and singular conic sections are illustrated in chapter 6.

	$\{V_Q\}$		p		$p \cup r$	
	D_{ext}	D_{in}	D_{ext}	D_{in}	D_{ext}	D_{in}
$AB \cap K = \emptyset$ $\overleftrightarrow{AB} \cap K \neq \{T\}$	$\{V_Q\}$	$\emptyset$	p	$\emptyset$	$\overrightarrow{S_pP} \cup \overrightarrow{S_rR}$	$\emptyset$
$AB \cap K = \emptyset$ $\overleftrightarrow{AB} \cap K = \{T\}$	$\{V_Q\}$	$\emptyset$	–	–	–	–
$AB \cap K = \{X_1, X_2\}$	–	–	–	–	–	–
$AB \cap K = \{T_0\}$	$\{V_Q\}$	$\emptyset$	–	–	–	–
$A = B$	$\{V_Q\}$	$\emptyset$	p	$\emptyset$	$\overrightarrow{V_QP} \cup \overrightarrow{V_QR}$	$\emptyset$

Table 3.1: The set D for singular conic sections.

	ellipse		parabola		hyperbola	
	D_{ext}	D_{in}	D_{ext}	D_{in}	D_{ext}	D_{in}
$AB \cap K = \emptyset$ $\overleftrightarrow{AB} \cap K \neq \{T\}$	1	$\emptyset$,1	1	$\emptyset$	2	$\emptyset$
$AB \cap K = \emptyset$ $\overleftrightarrow{AB} \cap K = \{T\}$	1	$\emptyset$	1	$\emptyset$	2	$\emptyset$
$AB \cap K = \{X_1, X_2\}$	$\emptyset$	1,2	$\emptyset$	1	1	1
$AB \cap K = \{T_0\}$	$\{T_0\}$	$\emptyset$,1	$\{T_0\}$	$\emptyset$	2	$\emptyset$
$A = B$	1	$\emptyset$	1	$\emptyset$	2	$\emptyset$

Table 3.2: The set D for regular conic sections. The numbers indicate the number of arcs in the set D.

Chapter 4

Boundary map

For the given points A, B, $X \in D \setminus D_e \subseteq K$ and the tangent line t at X to K, the Bézier curve b_{ACB} touching the conic section K is clearly identified. In order to find the middle control vertex C, we use the following map σ. This map is very similar to the map τ from the chapter 1, only the direction vector of the line t is expressed by the coefficients of the conic section K.

Definition 13 (Boundary map)**.** Let D be the set of points of contact for the given space-like points A, B and K. The map $\sigma : D \setminus D_e \to \mathbb{R}^2_1$ is called *boundary map* if for every $X \in D \setminus D_e$ holds $\sigma(X) = C$, for C from the definition 8, see the fig. 4.1.

Note 7. It is not possible to define the map σ on the points in D_e. If $T \in D_e \subset K$ then points A, B, T are collinear on the tangent line t to K. And there is an infinite number of points C such that Bézier curve b_{ACB} touches K in T. All suitable C form a half-line, therefore we are interested in the end point of the half-line, the point C_S.

Theorem 13. *Let the conic section $K \neq \{V_Q\}$ and $X = [x_0, y_0] \in D \setminus D_e$. Then the corresponding boundary map $\sigma : D \setminus D_e \to \mathbb{R}^2_1$ has the form*

$$\sigma(X) = \frac{b(t_0) - B^2_0(t_0)A - B^2_2(t_0)B}{B^2_1(t_0)} \;, \tag{4.1}$$

where $t_0 \in [0,1]$ *is a solution of the equation*

$$0 = \alpha t_0^2 + \beta t_0 + \gamma \tag{4.2}$$

and

$$\begin{aligned}
\alpha &= 2\left(\alpha_a - \alpha_b\right) \ , \\
\beta &= 2 \begin{pmatrix} x_0 & y_0 & 1 \end{pmatrix} \begin{pmatrix} 2k_A & 2k_B & k_1 \\ 2k_B & 2k_C & k_2 \\ k_1 & k_2 & k_3 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ 1 \end{pmatrix} \ , \\
\gamma &= -\frac{\beta}{2} \ ,
\end{aligned}$$

where

$$\begin{aligned}
\alpha_a &= \begin{pmatrix} a_x & a_y & 0 \end{pmatrix} \begin{pmatrix} k_A & k_B & k_D \\ k_B & k_C & k_E \\ k_D & k_E & k_F \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ 1 \end{pmatrix} \ , \\
\alpha_b &= \begin{pmatrix} b_x & b_y & 0 \end{pmatrix} \begin{pmatrix} k_A & k_B & k_D \\ k_B & k_C & k_E \\ k_D & k_E & k_F \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ 1 \end{pmatrix} \ , \\
k_1 &= \begin{pmatrix} k_A & k_B & k_D \end{pmatrix} \begin{pmatrix} -a_x \\ -a_y \\ 1 \end{pmatrix} \ , \\
k_2 &= \begin{pmatrix} k_B & k_C & k_E \end{pmatrix} \begin{pmatrix} -a_x \\ -a_y \\ 1 \end{pmatrix} \ , \\
k_3 &= 2 \begin{pmatrix} k_D & k_E & k_F \end{pmatrix} \begin{pmatrix} -a_x \\ -a_y \\ 0 \end{pmatrix} \ .
\end{aligned}$$

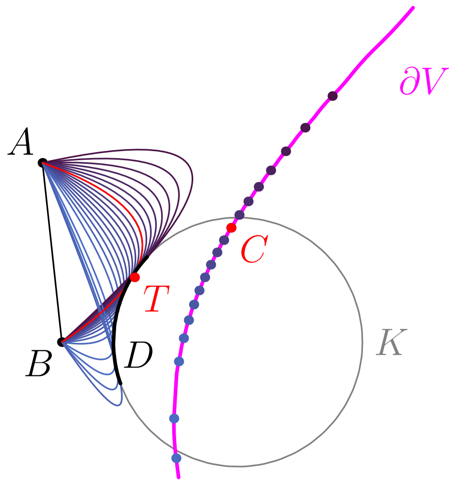


Figure 4.1: The boundary map σ, see that $\sigma(T) = C$.

Proof. Since we consider only real points $C \in \rho$, let $K = \{[x,y] \in \mathbb{R}^2 : k_A x^2 + 2k_B xy + k_C y^2 + 2k_D x + 2k_E y + k_F = 0\}$. Because of the point of the contact $X \in b_{ACB}(t)$, there exists $t_0 \in [0,1]$ such that $X = b_{ACB}(t_0) = B_0^2(t_0)A + B_1^2(t_0)C + B_2^2(t_0)B$. The point $X \in K$ is light-like so $X \notin \{A, B\}$ and $t_0 \notin \{0,1\}$. Because $X = [x_0, y_0] \in D \setminus D_e$ then holds $\left\langle \nabla f(x_0,y_0), \frac{d}{dt} b_{ACB}(t_0) \right\rangle = 0$. From this quadratic equation we obtain two roots t_0^0, t_0^1, but only one is in $(0,1)$. If both $t_0^0, t_0^1 \in (0,1)$ and $t_0^0 \neq t_0^1$ then $X \in D_e$. Let $t_0^0 \in (0,1)$. Then we substitute $t_0 = t_0^0$ into the Bézier curve equation and obtain relevant point C from the definition 8 for the point of the contact X. Hence, $C = \sigma(X)$. $\square$

Theorem 14. *For each $X \in D \setminus D_e$ there exists exactly one point C such that the Bézier curve $b_{ACB}(t) \cap K = \{X\}$.*

Proof. Let $t \in T_K$ touches the conic section in the point $X \in K$. We know, that the Bézier curve b_{ACB} has the points A, B as the end points, it contains the point X and it has the tangent line t at X. Hence, using the theorem 1, the quadratic b_{ACB} is uniquely determined up to the case $A, B, X \in t$ are collinear. But then $X \in D_e$. $\square$

Chapter 5

Area of admissible solutions

Lemma 15. *The boundary map σ is injective for the set $D \setminus D_e$ up to the points U_i.*

Proof. Let $X_1, X_2 \in D \setminus \{D_e \cup \{U_i\}\} \subset K$, $X_1 \neq X_2$ and $\sigma(X_1) = \sigma(X_2) = C$. Then, there exists a double contact Bézier curve b_{ACB} such that X_1, X_2 are the points of contact. Then $\{X_1, X_2\} = \{U_1, U_2\}$, which is a contradiction. $\square$

Note 8. Let the sharp arc $\{\overset{\frown}{T_1U_1} \cup \overset{\frown}{U_2T_2}\} \subset D_{in}$. The image $\sigma(\overset{\frown}{T_1U_1} - \{T_1\})$ is a connected curve l_1, because the boundary map σ is a continuous map. Also, the image $\sigma(\overset{\frown}{U_2T_2} - \{T_2\})$ is a connected curve l_2. There exists a point C_u such that the intersection $b_{AC_uB} \cap K = \{U_1, U_2\}$, hence, $C_u \in l_1$ and $C_u \in l_2$. Simultaneously, the points U_1, U_2 are the end points of the continuous arcs $\overset{\frown}{T_1U_1}$ and $\overset{\frown}{U_2T_2}$. Hence, the image of the sharp arc under the boundary map σ is the connected curve $l = l_1 \cup l_2$ (see the figure 6.4).

Now, we can find the corresponding point C for the points of contact from the set $D \setminus D_e$. The question is, how can we find the point C corresponding to a given point of contact $T \in D_e \subset D$.

Lemma 16. *Let the points A, B and $T \in D_e \subset K$ be collinear.*
(a) Let $AB \cap K = \{T\}$. The Bézier curve $b_{ACB} \cap K = \{T\}$ if and only if $C \in \overleftrightarrow{AB}$.
(b) Let $AB \cap K = \emptyset$. The Bézier curve $b_{ACB} \cap K = \{T\}$ if and only if $C \in \overrightarrow{C_SX} \subset \overleftrightarrow{AB}$, where $A, B \notin \overrightarrow{C_SX}$ and C_S is such that the derivative $\dot{b}_{AC_SB}(t_0) = 0$ for $T = b_{AC_SB}(t_0)$. For the special case $A = B$, then the point $C_S = A + 2(T - A)$.

Proof. Because A, B, T are collinear, the point $C \in \overleftrightarrow{AB}$. In the case (b), if C belongs to the opposite half-line to $\overrightarrow{C_S X}$, the Bézier curve and conic section have no common points. $\square$

Note 9. In the equation (4.1) of the boundary map σ, the inequality $B_1^2(t_0) > 0$ holds for $t_0 \in (0,1)$, so the map σ is continuous. The σ maps the connected set $D_{ext} \setminus D_e$ onto one connected curve l. If the point $T \in D_e \neq \emptyset$, then $\lim_{X \to T} \sigma(X) = C_S$ and the union $l \cup \overrightarrow{C_S X}$ is a connected curve.

Theorem 17 (Boundary of the set $V_\rho(A, B)$)**.** *The set of all feasible points C such that $b_{ACB} \cap K \subset D$, is the boundary of the set of admissible solutions $V_\rho(A, B)$. We denote it $\partial V_\rho(A, B)$.*

Proof. The point C belongs to the boundary of the set $V_\rho(A, B)$, if each neighbourhood N of the point C contains both the point $C_1 \in V_\rho(A, B)$ and $C_2 \notin V_\rho(A, B)$. So, we prove the existence of points $C_1, C_2 \in N$ such that $b_{AC_1B} \cap K = \emptyset$ and $b_{AC_2B} \cap K = \{X_1, X_2\}$ with transversed intersection (see fig. 5.1).

Let $A, B, \overline{u}$ as in the theorem 1 be fixed. Since $B_1^2(t_0) > 0$ for $t_0 \in (0,1)$, the map τ, which assign to each T the corresponding point C, is continuous. It means, for each neighbourhood N of the point C exists the neighbourhood M of the point T such that $\tau(M) \subset N$.

Let C be such that $b_{ACB} \cap K \subset D$ and $T \in b_{ACB} \cap K$. Let the line t with the direction vector $\overline{u}$ is the tangent line to K in T. From above, for arbitrary neighbourhood N of the point C exists the neighbourhood M of the point T such that $\tau(M) \subset N$. Let $T_1 = T + k\nabla f(T)$ and $T_2 = T - k\nabla f(T)$, where $k > 0$ is such that $T_1, T_2 \in M$ and they satisfies the requirements of the theorem 1. Let the lines t_1, t_2 be parallel to the line t (i.e. the vector $\overline{u}$ is their direction vector) and $T_1 \in t_1, T_2 \in t_2$. Then the points $C_1 = \tau(A, B, T_1, \overline{u})$ and $C_2 = \tau(A, B, T_2, \overline{u})$ are $C_1, C_2 \in N$. We obtain the Bézier curves b_{AC_1B}, b_{AC_2B}. Since each tangent line t_1, t_2 defines the supporting half-plane to the convex quadratic Bézier curve and the points A, B are space-like, it holds $b_{AC_1B} \cap K = \emptyset$ and $b_{AC_2B} \cap K = \{X_1, X_2\}$ with transversed intersection. $\square$

Note 10. We say, that the boundary of set of admissible solutions $\partial V_\rho(A,B)$ *is generated* by the set D mapped by the boundary map σ.

Lemma 18. *The boundary of admissible solutions $\partial V_\rho(A,B)$ consists of one or two continuous unbounded curves with degree at most four.*

Proof. The set D contains one or two arcs of the type D_{ext} or D_{in}. Due to the notes 8, 9, the boundary of admissible solutions consists of one or two continuous curves. They are unbounded, because the arcs in D are either unbounded or the point $T \in D_e$ generates the half-line. The degree of the curves is determined by the formula (4.1) of the map σ. $\square$

The curve $l \subset \partial V_\rho(A,B)$ divides the plane into two regions W_1, W_2 and one of them is the component of set of acceptable solution $V_\rho(A,B)$. The following theorem says which one.

Theorem 19. *Let $K \neq \{V_Q\}$ and $l \subset \partial V_\rho(A,B)$ is a connected curve. Let l divide the plane into two regions W_1, W_2. If l is generated by D_{ext}, then $W_i \subseteq V_\rho(A,B)$ if $A, B \in W_i$. If l is generated by D_{in}, then $W_i \subseteq V_\rho(A,B)$ if $A, B \notin W_i$.*

Note 11. In the case of hyperbola, each component generates one set of admissible solutions. Hence, we obtain the regions $V_1(A,B), V_2(A,B)$ for the K_1, K_2. For every point $C \in V_1(A,B)$, the Bézier curve $b_{ACB} \cap K_1 = \emptyset$. We are looking for the set of points C, such that $b_{ACB} \cap (K_1 \cup K_2) = \emptyset$. It holds for every point $C \in V_1(A,B) \cap V_2(A,B)$.

Note 12. In the case of $K = \{V_Q\}$, both $W_1, W_2 \subset V_\rho(A,B)$. Only for $C \in \partial V_\rho(A,B)$ is $b_{ACB} \cap K \neq \emptyset$, so $V_\rho(A,B) = \mathbb{R}^2_1 \setminus \partial V_\rho(A,B)$.

Finally, for the two space-like points A, B and the conic K the set of acceptable solutions $V_\rho(A,B)$ consists of one or two regions, see the table 5.1. It depends on the number of arcs in the set D and on the type of conic section.

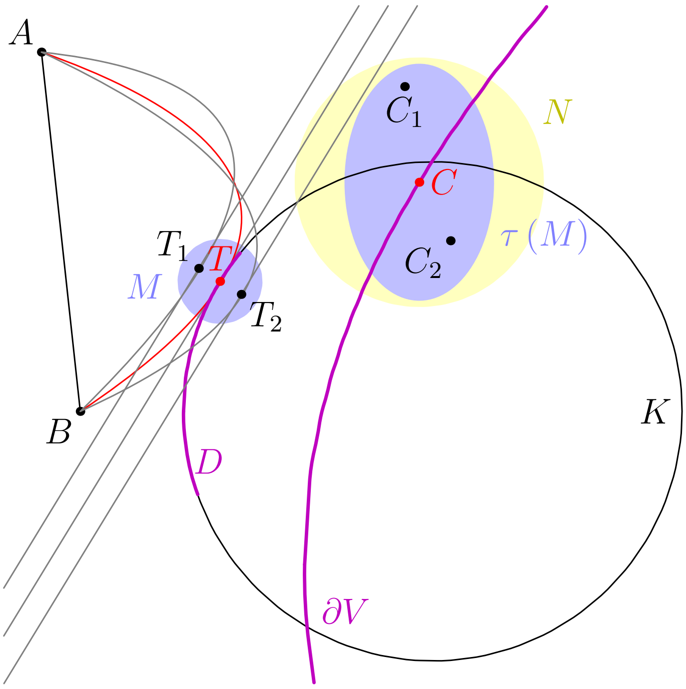


Figure 5.1: Let C be such that $b_{ACB} \cap K \subset D$ and $T \in b_{ACB} \cap K$. For arbitrary neighbourhood N of the point C, there exists a neighbourhood M of the point T such that for $T_1, T_2 \in M$ the corresponding $C_1 = \tau(T_1), C_2 = \tau(T_2) \in N$. Moreover, $b_{AC_1B} \cap K = \emptyset$ and $b_{AC_2B} \cap K = \{X_1, X_2\}$ with transversed intersection. Hence, $C \in \partial V_\rho(A, B)$.

	$\{V_Q\}$	p	$p \cup r$	ellipse	parabola	hyperbola
$AB \cap K = \emptyset$ $\overleftrightarrow{AB} \cap K \neq \{T\}$	2	1	1	1,2	1	1
$AB \cap K = \emptyset$ $\overleftrightarrow{AB} \cap K = \{T\}$	1	–	–	1	1	1
$AB \cap K = \{X_1, X_2\}$	–	–	–	1,2	1	1
$AB \cap K = \{T_0\}$	2	–	–	1,2	1	1
$A = B$	1	1	1	1	1	1

Table 5.1: The number of regions in the set of acceptable solutions $V_\rho(A, B)$.

Chapter 6

Examples

In this chapter, we present all generic cases of the mutual position of the segment AB and the conic section K.

The special group is the set of singular conic sections. There are figures 6.1, 6.2 for the examples of acceptable solutions for $K = \{V_Q\}$ and isotropic lines as the conic section K.

For the examples of acceptable solutions $V_\rho(A, B)$ for ellipse (unit circle) see the figures 6.3, 6.4, 6.5. The figure 6.6 shows the acceptable solutions for parabola and the figures 6.7, 6.8 shows the same for hyperbola.

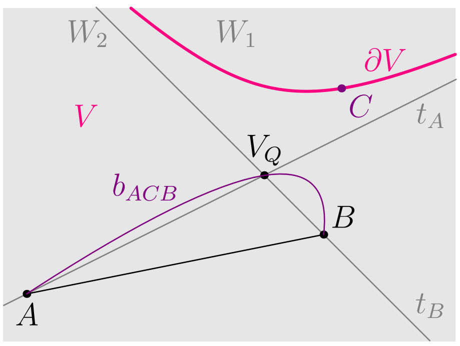
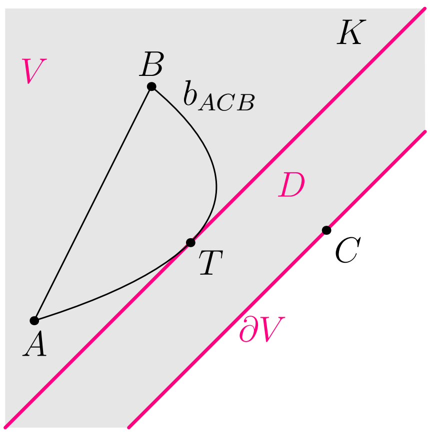
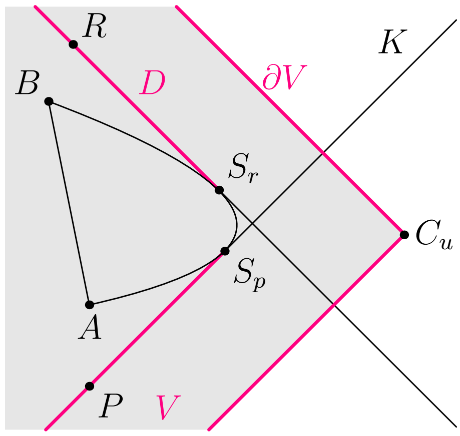


Figure 6.1: Let the conic section $K = \{V_Q\}$. There is only one possible point of contact, so $D = \{V_Q\}$. The corresponding boundary of admissible solutions $\partial V_\rho(A, B)$ divides the plane ρ into two regions $W_1, W_2 \subset V_\rho(A, B)$, because that $b_{ACB} \cap K \neq \emptyset$ holds only for $C \in \partial V_\rho(A, B)$. So, the set of admissible solutions $V_\rho(A, B) = \mathbb{R}^2_1 \setminus \partial V_\rho(A, B)$.



(a)



(b)

Figure 6.2: (a) Let the conic section $K = p$. If A, B lie in the same open half-plane generated by K, the set of exterior points of contact $D_{ext} = p$, else $D_{ext} = \emptyset$. The set of interior points of contact is always $D_{in} = \emptyset$. The set $D = p$ generates the boundary $\partial V_\rho(A, B)$. This boundary is the parallel line with the line p, because the coefficient α in the equation (4.2) is *constant* (it does not depend on the point of contact). The set of admissible solution $V_\rho(A, B)$ consists of one region. Since the boundary is generated by the exterior points of contact, $A, B \in V_\rho(A, B)$.

(b) Let the conic section $K = p \cup r$. The points A, B have to lie in the same quadrant with respect to K. Let it be the quadrant determined by two half-lines $\overrightarrow{V_Q P} \subset p$ and $\overrightarrow{V_Q R} \subset r$. If there exists the Bézier curve b_{AC_uB} with double contact $M = \{S_p, S_r\}$ with K, the set of exterior points of contact $D_{ext} = \overrightarrow{S_p P} \cup \overrightarrow{S_r R}$. Else, it holds $S_p = S_r = V_Q$ and $D_{ext} = \overrightarrow{V_Q P} \cup \overrightarrow{V_Q R}$. The set of interior points of contact is always $D_{in} = \emptyset$. The boundaries generated by the half-lines $\overrightarrow{S_p P}, \overrightarrow{S_r R}$ are connected due to the fact that $\sigma(S_p) = \sigma(S_r) = C_u$. Hence, the set of admissible solution $V_\rho(A, B)$ contains of one region. Because the boundary is generated by the exterior points of contact, $A, B \in V_\rho(A, B)$.

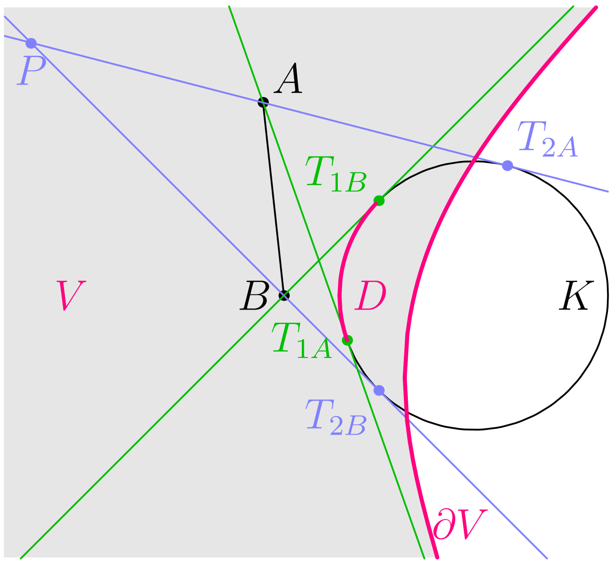


Figure 6.3: According to the theorem 9, the set $D_{ext} = \wideparen{T_{1A}T_{2A}} \cap \wideparen{T_{1B}T_{2B}}$. The set $D_e = \emptyset$, because $\overleftrightarrow{AB} \cap K = \emptyset$. The set $D_{in} = \emptyset$, because the point P lies in the opposite half-plane generated by $\overleftrightarrow{AB}$ as the points T_{2A}, T_{2B} (corresponding points to the tangent lines $t_{2A}, t_{2B} \notin T_{sep}(AB, K)$). Hence, the set $D = D_{ext}$ generates the boundary of admissible solutions $\partial V_\rho(A, B)$. Because the boundary is generated by the exterior points of contact, the $A, B \in V_\rho(A, B)$ according to the theorem 19. The set of admissible solutions consists of one connected region.

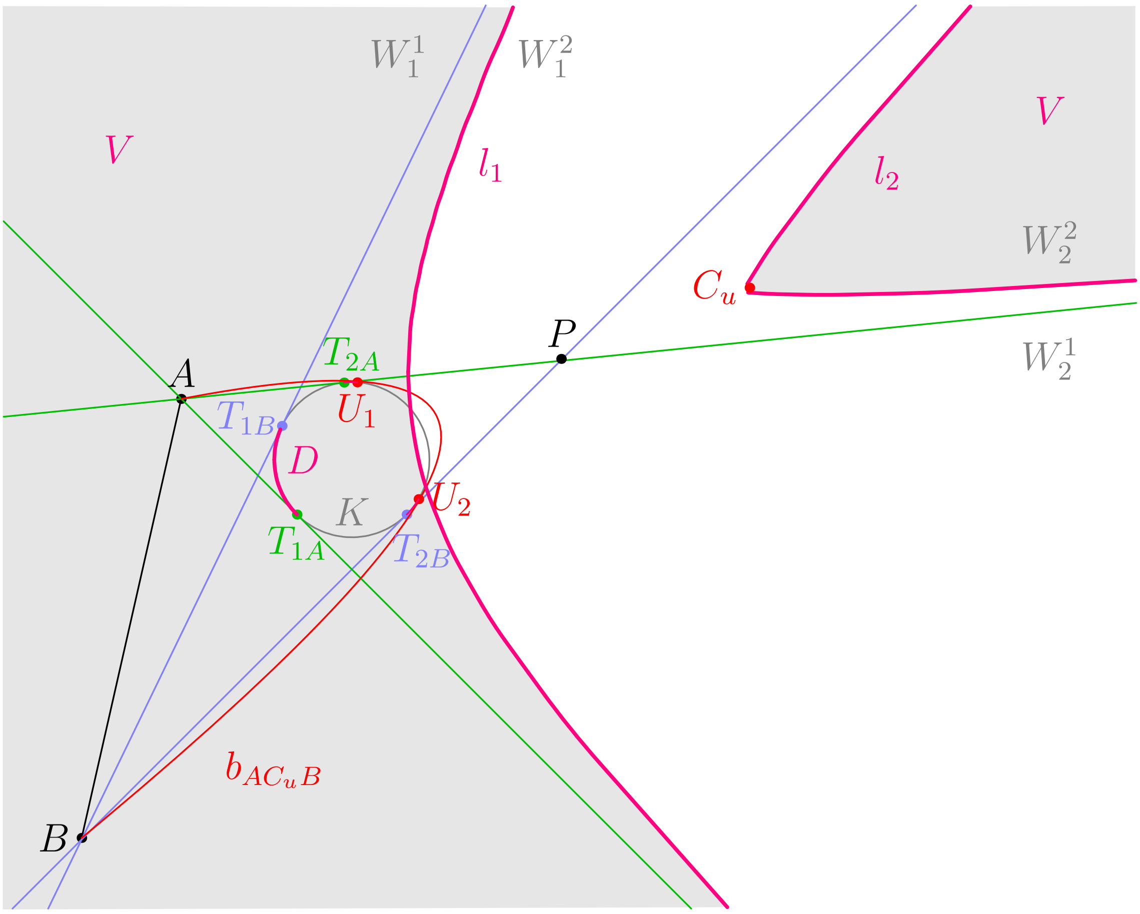


Figure 6.4: The set $D_{ext} = \overset{\frown}{T_{1A}T_{2A}} \cap \overset{\frown}{T_{1B}T_{2B}}$. The set $D_{in} \neq \emptyset$, because the point P lies in the same half-plane generated by $\overleftrightarrow{AB}$ as the points T_{2A}, T_{2B}. The set D_{in} is the sharp arc $\overset{\frown}{T_{2A}U_1} \cup \overset{\frown}{U_2T_{2B}}$. The split of the arc is caused by the existence of the curve b_{AC_uB}, which has double contact with the conic section K. As one can see, $b_{AC_uB} \cap K = \{U_1, U_2\}$. Therefore, the set of points of contact $D = D_{ext} \cup D_{in}$ generates the curves l_1, l_2 such that $l_1 \cup l_2 = \partial V_\rho(A, B)$. Because the curve l_1 is generated by the exterior points of contact, the subset of the $V_\rho(A, B)$ is such that region W_1^1 that $A, B \in W_1^1$. The curve l_2 is generated by the interior points of contact, so the subset of the $V_\rho(A, B)$ is such that region W_2^2 that $A, B \notin W_2^2$. The set of admissible solutions $V_\rho(A, B) = W_1^1 \cup W_2^2$ consists of two regions.

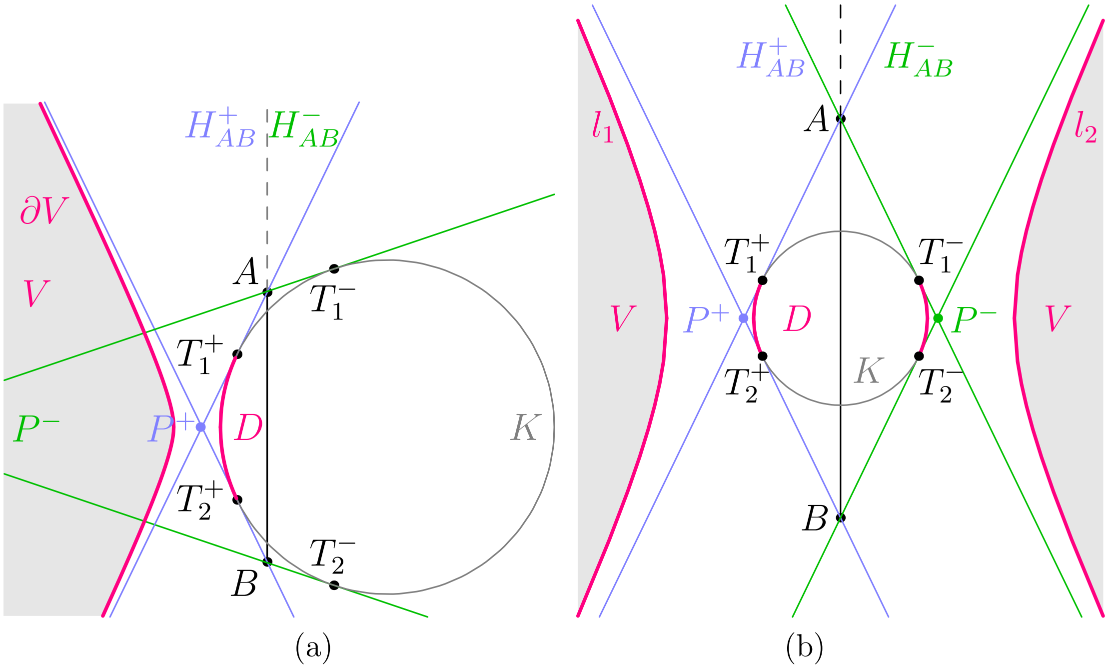
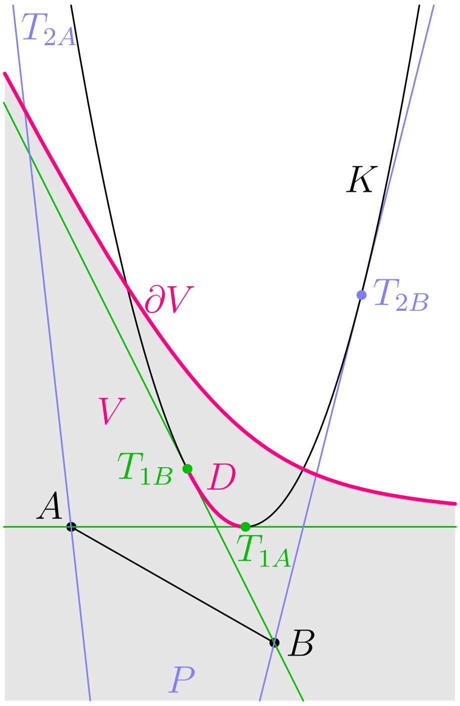
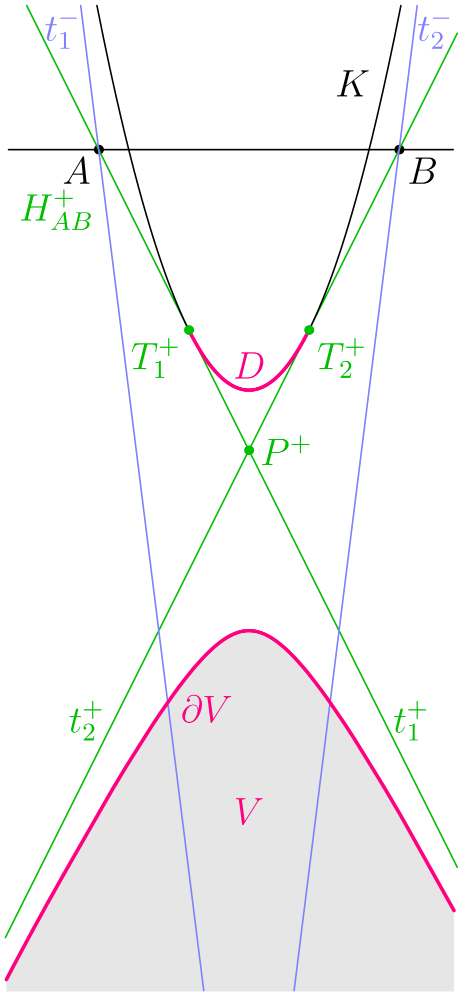


Figure 6.5: (a) The $AB \cap K \neq \emptyset$, so according to the theorem 9, there holds $D_{ext} = \emptyset$. We divide the non separating tangent lines from the points A, B into two pairs due to the position of their corresponding points of contact in the half-planes H^+_{AB}, H^-_{AB}. As one can see, the point $P^- \in H^+_{AB}$ and the points $T_1^-, T_2^- \in H^-_{AB}$. It means, the tangent lines t_1^-, t_2^- diverge in H^-_{AB}, so the set $D^-_{in} = \emptyset$ according to the theorem 11. The pair of the tangent lines t_1^+, t_2^+ converge in H^+_{AB}, so the set $D^+_{in} = \overset{\frown}{T_1^+ T_2^+}$ assuming the hypothesis 1. Hence, the set of points of contact $D = D^+_{in}$ generates the boundary of admissible solutions $\partial V_\rho(A, B)$. The set of admissible solutions $V_\rho(A, B)$ consists of one region.

(b) In this case, both pairs of tangent lines t_1^-, t_2^- and t_1^+, t_2^+ converge in corresponding half-planes, both sets $D^-_{in}, D^+_{in} \neq \emptyset$. Then, the set of points of contact $D = D^-_{in} \cup D^+_{in}$ generates two curves l_1, l_2. Hence, the boundary of admissible solutions $\partial V_\rho(A, B) = l_1 \cup l_2$ determines two components of $V_\rho(A, B)$. The boundary of admissible solutions $\partial V_\rho(A, B)$ is generated by the interior points of contact, so $A, B \notin V_\rho(A, B)$.



(a)



(b)

Figure 6.6: (a) Because $AB \cap K = \emptyset$, the set of exterior points of contact $D_{ext} = \overset{\frown}{T_{1B}T_{1A}}$ is determined by the pair of tangent lines $t_{1B}, t_{1A} \in T_{sep}(AB, K)$. The pair of non separating tangent lines t_{2A}, t_{2B} diverge (the points T_{2A}, T_{2B} and the point P lying in the opposite half-plane due to the line $\overleftrightarrow{AB}$). Therefore, the set of interior points of contact $D_{in} = \emptyset$. The set of points of contact $D = D_{ext}$ generates the boundary of admissible solutions $\partial V_\rho(A, B)$. The set of admissible solutions $V_\rho(A, B)$ consists of one region.

(b) If $AB \cap K = \{X_1, X_2\}$, one pair of non separating tangent lines always diverge and the rest pair always converge. Without loss of generality, the set of points of contact $D = D_{in}^+ = \overset{\frown}{T_1^+ T_2^+}$. The set of admissible solutions $V_\rho(A, B)$ consists of one region.

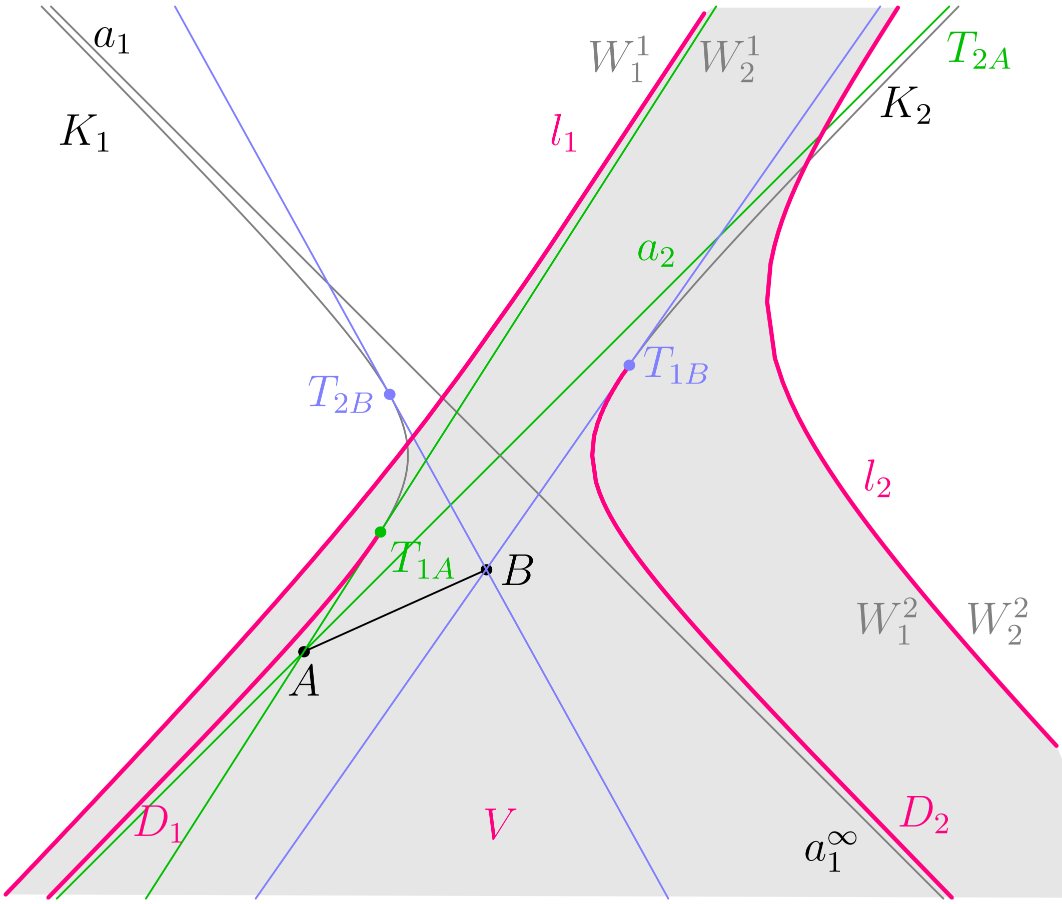


Figure 6.7: Let us analyse the component K_1. Since $T_{2A} = a_2^\infty$, the set $S(A, K_1) = \overset{\frown}{T_{1A}a_2^\infty}$, where both points $T_{1A}, a_2^\infty \in \{K_1 \cup a_1^\infty \cup a_2^\infty\}$. The point $T_{2B} \in K_1$, but the point $T_{1B} \notin K_1$. Hence, we have to substitute the point T_{1B} by $a_2^\infty \in \{K_1 \cup a_1^\infty \cup a_2^\infty\}$, because the asymptote $a_2 \in T_{sep}(B, K_1)$. Then, the set $S(B, K_1) = \overset{\frown}{a_2^\infty T_{2B}}$. According to the theorem 9, the set $D_{ext}^1 = \overset{\frown}{T_{1A}a_2^\infty} \cap \overset{\frown}{a_2^\infty T_{2B}}$. The set $D_{in}^1 = \emptyset$. The set of points of contact $D^1 = D_{ext}^1$ generates the curve $l_1 \subset \partial V_\rho(A, B)$. The curve l_1 divides the plane ρ into two regions W_1^1, W_2^1. Let the points $A, B \in W_2^1$. According to the theorem 19, the set of admissible solutions for the component K_1 is $V^1(A, B) = W_2^1$.
Now, let us analyse the component K_2. We have to substitute the point T_{1A} by the point $a_1^\infty \in \{K_2 \cup a_1^\infty \cup a_2^\infty\}$. The set $S(A, K_2) = \overset{\frown}{a_1^\infty T_{2A}} = K_2$. Similarly, we have to substitute the point T_{2B} by the point $a_1^\infty \in \{K_2 \cup a_1^\infty \cup a_2^\infty\}$. The set $S(B, K_2) = \overset{\frown}{a_1^\infty T_{1B}}$. Hence, the set $D_{ext}^2 = \overset{\frown}{T_{1B}a_1^\infty}$. The set $D_{in}^2 = \emptyset$. The set of points of contact $D^2 = D_{ext}^2$ generates the curve $l_2 \subset \partial V_\rho(A, B)$ and if the points $A, B \in W_1^2$, then $V^2(A, B) = W_1^2$.
Finally, the set of admissible solutions for the conic section K is $V_\rho(A, B) = V^1(A, B) \cap V^2(A, B)$.

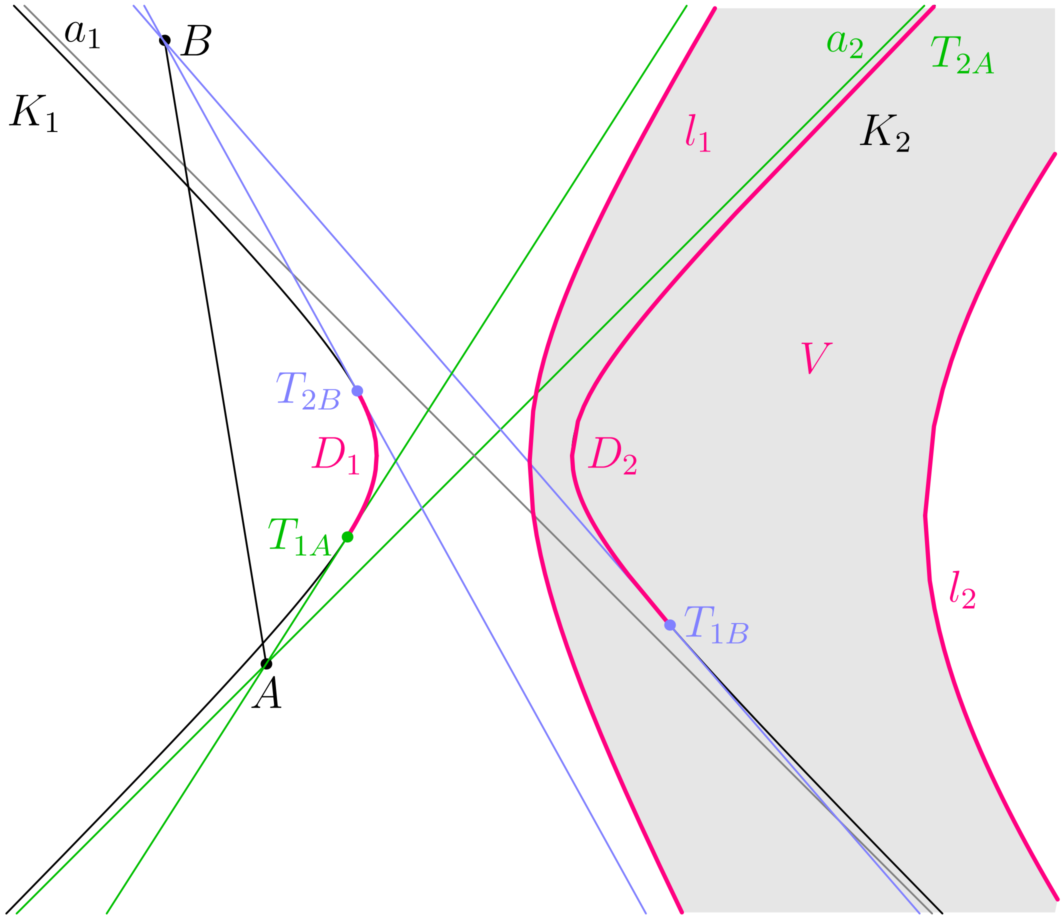


Figure 6.8: For the component K_1, the set of exterior points of contact $D^1_{ext} = \emptyset$ and the set of interior points of contact $D^1_{in} = \overset{\frown}{T_{1A}T_{2B}}$. Then the curve l_1 generated by D^1 determines the region $V^1(A,B)$ such that $A, B \notin V^1(A,B)$.
For the component K_2, since $T_{2A} = a_2^\infty$, the set of exterior points of contact $D^2_{ext} = \overset{\frown}{T_{1B}a_2^\infty}$ and the set of interior points of contact $D^2_{in} = \emptyset$. Then the curve l_2 generated by D^2 determines the region $V^2(A,B)$ such that $A, B \in V^2(A,B)$.
Finally, the set of admissible solutions for the conic section K is $V_\rho(A,B) = V^1(A,B) \cap V^2(A,B)$.

Chapter 7

Conclusion

For the given points A, B in three dimensional Minkowski space, we described the area $V(A, B)$ of all such points C that the quadratic Bézier curve with control points A, C, B is space-like.

For this purpose, we define the necessary conditions for the end points A, B of the Bézier curve. Then, we fixed the points A, B and we looked for the admissible points C. We solved the problem for each type of conic section K, which the plane ρ containing the points A, B cuts in the light cone Q.
The area $V_\rho(A, B)$ containing the admissible points $C \in \rho$ is described via method of contact point. We proved, that the boundary of this area $\partial V_\rho(A, B)$ is determined by such points C that b_{ACB} touches to the conic section K. Hence, we found the set of admissible points of contact D for the given points A, B as union of exterior and interior points of contact. Then, we showed how the set D determines boundary $\partial V_\rho(A, B)$ using the boundary map σ.
We have illustrated this study with typical examples, which show the area $V_\rho(A, B)$ for each type of conic section K.

When the hypothesis 1, 2 are solved, then the case of quadratic space-like curves will be completed. We have indications leading to the proves, they still must be elaborated.

Project of Dissertation

In this chapter, we present the plan of our future work. We want to focus on conclusion of space-like conditions for quadratic curves. In the rest of our study, we will look for the space-like condition of the Bézier curves of higher degree in three dimensional Minkowski space. We start to study a conditions for cubic curves, using the knowledge and methods from quadratic case as much as possible. Additional methods are to be applied too.

Future research

We would like to proof the hypothesis 1, 2 on the set of interior points of contact. Then, the space-like conditions for the control points of the quadratic curve will be entirely completed.
In order to prove the hypothesis 1, we use that if the pair of tangent lines t_1^+, t_2^+ converge in the half-plane H_{AB}^+, then the lines v_1, v_2 from the points A, B, which are orthogonal to the segment AB, have no common points with K in the half-plane H_{AB}^+. Let the lines v_1, v_2 determine the point at infinity v^∞. In order to $D_{in}^+ \neq \emptyset$, there have to exists the Bézier curve $b(t) \subset \overline{H_{AB}^+}$ such that $b(t) \cap K = \emptyset$. We use as "the limit case" that the Bézier curve $b_{Av^\infty B} \cap K = \emptyset$.
Regarding the hypothesis 2, let the quadratic Bézier curve $b_{ACB}(t)$ be a part of the parabolic arc p_C. We denote the curvature of the curve p_C at the point $X \in p_C$ by $\kappa(p_C, X)$. Let the point $P_C \in b_{ACB} \subset p_C$ is such that for arbitrary $X \in b_{ACB} \subset p_C$ holds $\kappa(p_C, P_C) \geq \kappa(p_C, X)$. Let the sharp arc $\{T_1\overset{\frown}{U_1} \cup U_2\overset{\frown}{T_2}\} \in D_{in}$ generate the connected curve $l \in \partial V_\rho(A, B)$. For verification of the hypothesis 2, we use

the following hypothesis. The point $C_u \in l$ is such that for arbitrary $C \in l$ holds $\kappa\left(p_{C_u}, P_{C_u}\right) \leq \kappa\left(p_C, P_C\right)$.
After the proof of both hypothesis, the set of admissible solutions will be found for the each type of conic section in the plane ρ.

Until now, we illustrate the situation only in the plane. Hence, we would like to make a 3D visualisation of the situation. We will fix the points A, B and we will consider a several positions of the plane ρ. We present a model of set of all admissible solutions to the problem describing its surface boundary. We detect its singularities.

The analytic expression of the boundary of admissible solutions in the plane ρ is another interesting question. In order to find it, we express all definitions and propositions using in this work by analytical equations and inequalities. Also, we suppose that the parameter t for the points of contact changes continuously. We look for the interval $J \subset [0, 1]$ of the parameter t, and the expression of the boundary from the equation (4.1) of the boundary map σ.
This generates the question, what kind of the object is the boundary in 3D? If we know the analytic expression of the boundary in the plane, we parametrize it to find the analytic expression of the boundary of the "spatial set of admissible solutions". Clearly, the same approach can be used in arbitrary dimension for the case of quadratic Bézier curves.

While looking for the set of points of contact $D \subset K$, we use the term point at infinity. Then, in the case of K as hyperbola or parabola, our illustrations are missing the points of the contact in the infinity. Inspired by the books on projective geometry [Far99] and [Bix06], we would like to construct a spherical model of the projective plane in order to compare the number of regions in the set of acceptable solutions $V_\rho(A, B)$ with the current results. We suppose, the classification of all possible cases for regular conic sections unifies and simplifies.

After the conclusion of the quadratic case, we proceed to find analogous conditions for the control points of cubic Bézier curves. The situation becomes much more complicated, because cubic curves are uniquely determined by four control points. In the past, we used the fact that quadratic curve lies in the plane. Hence, we will consider the planar cubics first. We fix the three control points and we will try to find (in the plane ρ) the set of admissible solutions for the remaining control point. We look for analytic conditions, which are visualized (e.g. by *Asymptote*).

An interesting topic covers considering rational curves in Minkowski space. Quadratic case will be searched from the projective geometry point of view. The general conditions for higher degree curves will be difficult to grasp. Hence, we consider certain classes such as rational curves with positive weight, convex curves and others to gain insight into the structure of space-like curves.

In the case of general cubic curve, we plan to use the fact that the derivative of cubic curve is a quadratic function although it might not be space-like. With the fact that the end control points have to be space-like, we will try to get necessary conditions for the control points or its linear combinations. If these are sufficient conditions too, we verify it experimentally at first and theoretically later. We have a hypothesis, that if the derivative is space-like (definition 1) along the curve and end point is space-like, then the whole curve is space-like (definition 2). This leads to certain class of cubic space-like curves putting together the results gained so far.

We try to combine the curve results to surface case for tensor-product surfaces.

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