Fractals

Part 8 : Stochastic fractals
Why randomness?

- Generalized set of shapes
- More nature-like
- Not strict self-similarity
- Often using Brownian motion
- For each type of fractal
- Using randomness in any stage
Classical fractals - Koch
Sierpinski
Percolation

- Given triangular or square lattice
- Given probability $p$
- Color each sub cell with probability
- Check number of disjunctive parts
- Many -> one clusters = percolation
- $p_c$ – percolation threshold
Percolation 2

$p = 0.3$

$p = 0.4$

$p = 0.6$

$p = 0.7$
Forest fire simulation

- Square lattice
- Trees with probability $p$
- For $p>p_c$ whole forest will burn
- For $p<p_c$ only part of forest will burn
- For $p=p_c$ the forest will burn for longest time
- $p_c \approx 0.5928$
Forest fire simulation 2
Renormalization

- For triangular lattice
- Sites -> super-sites
- Super-site is occupied if two or three sites are occupied
Renormalization 2

- With $p' > p$ we fill gaps.
- With $p' < p$ clusters will vanish.
- $p' = p$ we expect similarity.
- $p' = p^3 + 3p^2(1-p)$
- $p = p_c = 0.5$
- Statistical self-similarity.
Renormalization 3
Particles aggregation

- Laboratory experiment
- Zinc-metal leaves
DLA

- Particle is moving with Brownian motion
- If free particle approaches to sticky particle, it stops and becomes sticky
- Repeating with another particle
- Simulation using pixels
- Diffusion Limited Aggregation
Simulating DLA
Problems

- What is the fractal dimension?
- Density of particles decreases from center. Is there power law for it?
- Is voltage with relation to fractal dimension?
- Is size of aggregate with relation to fractal dimension?
Problems 2

- Still not precise solutions
- $D \sim 1.7$
Other DLAs
3D DLA

local.wasp.uwa.edu.au/~pbourke/fractals/dla3d/
Using DLA & percolation

- Distribution of galaxies
- Microcosm
- Porous media
- Clouds, rainfall areas
- Simulation of growth
- Crystals
Brownian motion

- “Chaotic” movement of particles
- Related to Gaussian distribution
- Statistically self-similar fractal
- Base for other statistical fractals
Gaussian distribution

- **Probability density function**

\[ f(x; \mu, \sigma^2) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \]

- **Cumulative distribution function**

\[ \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{x}{\sqrt{2}} \right) \right], \quad x \in \mathbb{R}. \]

\[ F(x; \mu, \sigma^2) = \Phi \left( \frac{x-\mu}{\sigma} \right) = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{x-\mu}{\sigma \sqrt{2}} \right) \right], \quad x \in \mathbb{R}. \]
Simulation of BM

- 1D simulation – Wiener process
  - \( W_0 = 0, \ W_t-W_s \sim N(0, t-s) \)
- Proceeding in \( t \) with uniform steps
  - \( X(0)=0, \ X(k)=D_1+...+D_k \ ; \ k=1,2,3,... \)
  - Using Gaussian random numbers (expected value 0, variance 1) as displacement \( D_k \)
- Scaling:
  - \( V_t=1/\sqrt{c}W_{ct} \) is another Wiener process
  - in \( x \) 2 times, in \( y \sqrt{2} \) times
  - \( X(t+dt)-X(t) \) and \( 1/r^{0.5}(X(t+r.dt)-X(t)) \) are statistically equivalent
- \( X(t+dt) = X(t) + v \cdot dt^{0.5} \cdot N(0,1) \)
  - \( v \) – speed of particle
Simulation of BM
Midpoint displacement

- Construction of parabola $P(x) = a - bx^2$, $b > 0$
- Archimedes
Midpoint displacement

\begin{equation}
T_w(x) = \sum_{n=0}^{\infty} w^n s(2^n x)
\end{equation}

\begin{equation}
\text{blanc}(x) = \sum_{n=0}^{\infty} \frac{s(2^n x)}{2^n}
\end{equation}

\begin{equation}
s(x) = \min_{n \in \mathbb{Z}} |x - n|
\end{equation}

Takagi curve

Landsberg curve
Midpoint displacement

- In each stage for each line displace midpoint in y-direction with Gaussian random number multiplied by scale

- $X(0)=0; \ X(1)=\text{GRN}$

- $X(1/2)=(1/2)*(X(0)+X(1))+D_1/\sqrt{2}$

- Recursive algorithm
Hurst exponent

- Creating Fractal Brownian Motion (FBm)
- Can be generalized with parameter $H$
- In $i$-th step, multiply Gaussian random number by $2^{-Hi}$
- $H$ – Hurst exponent, $0 < H < 1$
- Curves dimension $D = 2-H$
- Surfaces dimension $D = 3-H$
Exponents and dimensions

$H = 0.2$
$D = 1.8$

$H = 0.5$
$D = 1.5$

$H = 0.8$
$D = 1.2$
Midpoint displacement 2
Midpoint displacement ext.

- Modeling natural objects
- Coastline = initial closed polygon,
- Landscape = extension to 2D, dividing triangles or squares
- Fake clouds = colored height map
- True clouds = map of points above threshold
Coastline
Landscapes

- Better method – subdividing square
- Diamond-square algorithm
Improvements

- Merging of different terrains with several Hurst exponents
- Adding fractal noise to smooth terrain
- Spectral Synthesis Method – remove high frequencies using Fourier transform
Visualization

- Mostly visualization of height map
- Coloring: Mapping aerial textures, using height for color
- Acceleration structures – quadtrees
- HW support – tessellation, vertex shaders – displacement mapping
- Raytracing – subdivided are only parts of terrain that are hit by some ray
Landscapes
Professional landscape
Generating clouds

- 2D clouds = height map
- Draw the height field as color map with different transparency based on height
End of Part 8